Abstract: The availability of ICT offers opportunities to reconsider teaching and learning in the calculus curriculum. In this paper, some potential contributions of one form of ICT, the hand-held graphics calculator, are described and evaluated. Although algebraic calculators, graphics calculators with symbolic manipulation capabilities, have been available for some years now, attention in the paper is restricted to calculators without algebraic capabilities. These are more likely to be available on a wide scale in many East Asian countries than are algebraic calculators, and significant experience with them in schools has now accumulated. The paper will consider a number of key calculus concepts, such as the derivative of a function at a point, the derivative function, continuity, asymptotic behaviour, convergence, limits, integration and differential equations, to develop the argument that ICT offers an opportunity to help students understand the concepts behind the calculus, upon which standard techniques and symbolic procedures depend. A quality mathematics education needs to focus attention on these key concepts, and the paper demonstrates how ICT offers new ways of doing this successfully. Work is needed to take advantage of these opportunities in educationally effective ways.

Keywords: calculus, graphics calculator, concepts, teaching, learning

INTRODUCTION

Calculus at the secondary school level has traditionally represented the peak of school mathematics, and in many cultures has been available only to the most capable students. Until recently, many calculus curricula have focused on developing standard techniques, such as those concerned with differentiation and integration, with an emphasis on symbolic procedures for carrying these out in a range of situations. These characteristics have been reinforced by external assessment agencies such as examination boards.

Over the short lifetime (since 1998) of the East Asian Regional Conferences on Mathematics Education, it is clear that electronic technology has become a major influence on school mathematics, with impacts felt in all communities. As a consequence, each EARCOME has included significant presentations related to technology in school mathematics. Technology is omnipresent in East Asian communities, especially in cities, indicating that these changes are of global significance and are not confined to western industrialised and affluent nations.

This paper is concerned with exploring the possible ways in which one kind of technology, the graphics calculator, might be productively used by calculus students and their teachers. The paper provides an analysis to describe some affordances provided by graphics calculators: opportunities for significant changes to the teaching and learning of calculus. It is recognised that access to such affordances is not by itself sufficient to bring about changes in practice, with various issues associated with this briefly referred to in the final part of the paper.
THE GRAPHICS CALCULATOR

The choice of a graphics calculator is motivated by a view that this represents the best prospect for providing widespread access to technology for mathematics within typical schools in the East Asian region. This view was elaborated in detail in Kissane (1995; 1998; 2005) and so will not be repeated in detail here. The main parts of the argument are that graphics calculators are relatively inexpensive, very portable and include significant educationally valuable software. In contrast, other forms of technology have significant barriers to widespread access. Computers are significantly more expensive than calculators, often depend on easy access to computer laboratories, require relatively expensive computer software, are unlikely to be available widely enough to be taken into account by curriculum developers and are very unlikely to be permitted for use in important examinations. The Internet, promising in many ways, suffers most of the limitations of computers, and has the additional problem that relatively little good software is freely available for student use.

This is not to say that the graphics calculator is an optimum choice; in the absence of the constraints of the real educational world, computers have the potential to provide a more powerful environment, enriched by large, fast and colourful screens and supported by powerful and multi-purpose software. However, the graphics calculator continues to represent the best compromise between the latest technology and what is likely to be manageable for a nationwide school system with necessarily limited financial resources.

A consequence of these arguments is that the graphics calculator in 2007 continues to be the most likely way for senior secondary school students and their teachers to integrate the use of technology into the teaching and learning program for all. Recent developments have made it clear that both Singapore and Malaysia have recognised these advantages, in the same way that most Australian states now have. Indeed, it is now more than a decade since the US College Board approved the use of graphics calculators on the Advanced Placement Calculus examinations, also recognising the educational advantages of doing so (Kennedy, 2002).

Although experience is now accumulating with, and there is much interest in, the place of graphics calculators with Computer Algebra System (CAS) capabilities, this paper is restricted to graphics calculators that do not include significant capabilities of this kind.

CONCEPT REPRESENTATION ON GRAPHICS CALCULATORS

Although there is an emphasis on procedures in calculus in some classrooms, and an unavoidable emphasis on formal procedures in examinations, the concepts of calculus are the most important elements, and are the focus of this paper. Procedural competence can best be developed when students understand the underlying ideas well. In this section of the paper, we consider how some key concepts of the calculus are represented by graphics calculators, in order to illustrate how educational experiences might be designed differently by teachers than in situations for which technology is not available. Space precludes a complete treatment of possibilities, many of which are described in some detail in Kissane & Kemp (2006). A recent graphics calculator, the Casio fx-9860G, is used for all of the examples below.

Local Linearity

A key concept underpinning the idea of the derivative of a function at a point is that of local linearity: that, on a small enough interval, the graph of a differentiable function can be well approximated by a line. Maschietto (2004) described productive work in classrooms using this powerful idea, accessed through a process of ‘zooming in’ on
graphs on calculators. Figure 1 shows the beginning of an example, which can be readily continued by repeatedly zooming in to produce increasingly good approximations to a line around $x = 1$, although it is clear at first that the graph is parabolic and hence is curved.

![Figure 1: Successive approximations to local linearity of $y = x^2$ at $x = 1$ by zooming in.]

Numerical Derivative at a Point

The concept of a derivative at a point involves the notion of a limit, which is notoriously difficult for beginning students (and was not in fact developed formally in the early years of the calculus by Newton and Leibniz, but appeared instead many years later). A conceptually easier idea involves a rate of change over a small interval, provided by the numerical derivative command of graphics calculators. This concept is closely related to that of local linearity, as it represents the slope of the (local) line. Figure 2 shows an example of the calculator’s affordance here: as students trace the graph of the function, both the coordinates of each point and the numerical derivative at the point are provided. A significant conceptual advantage of this image is that it allows for the idea of the rate of change of the function itself, rather than that of tangents to the function, described briefly in the next section.

![Figure 2: Numerical derivatives of $y = x^2 - x - 1$ obtained by tracing]

Similarly, numerical derivatives can be obtained in a table of numerical values of a function. In the example shown in Figure 3, it is clear that the derivatives are steadily increasing as $x$ increases, and the idea of the derivative itself being a function is suggested by the table.

![Figure 3: Numerical derivatives of $y = x^2 - x - 1$ obtained in a table of values]

Using these capabilities, students can explore many aspects of the relationships between functions, graphs and derivatives, laying important groundwork for a more formal symbolic study of these at a later stage.

Tangents

Traditional approaches to the idea of a derivative at a point have involved a secant that eventually becomes a tangent in the limiting case of the two end-points of the secant coinciding. On a graphics calculator, tangents can be drawn at a point, allowing for a version of this concept image to be experienced by students, and also allowing for a line with a particular slope to be seen (rather than just the numerical value of the slope). Figure 4 illustrates this, using the same function and points as for
Figure 2. In this case (because the derivative trace of the calculator is turned on), the screen shows both the tangents at the chosen points as well as their equations. These images seem likely to help students connect the rate of change of the function at a point with the tangents to the curves at the point.

Figure 4: Tangents to \( y = x^2 - x - 1 \) obtained by tracing

Derivative Functions

As suggested above, the concept of a derivative function is a natural extension of the idea of a derivative of a function at a point: a derivative function describes the entire family of derivatives at a point for a particular function. On a graphics calculator, the standard symbol for a derivative is used (dy/dx), and both a graphical and a numerical representation of the derivative function can be obtained for any function.

Figure 5: Representing a derivative function graphically and numerically

These representations, linking functions and their derivatives through simultaneous displays, offer many opportunities for classroom exploration. For example, student attention can be drawn to the significant features of a graph (eg slope, turning points) and the associated values of the derivative function. There are possibilities for individual, small group and whole class work (the latter using a public display screen), with good activities available through making changes to the original function, defined above as Y1. For example, the case shown in Figure 5 suggests that the derivative function is linear for the chosen quadratic function, supported both graphically and numerically; student exploration will reveal that this is the case for all quadratic functions, not only this one.

Optimisation

Historically, a major motivation for differential calculus involves optimising situations, through the location of relative maxima and minima of functions. At first sight, this might seem to be undermined when students have access to a graphics calculator which allows for very good approximations to these points to be obtained visually, or numerically, using maximum and minimum commands, as suggested by Figure 6.

Figure 6: Obtaining relative maxima and minima from a graph or numerically

While students may well encounter these ideas earlier than the calculus, through the agency of graphics calculators, the formal study of the calculus might be seen nowadays as an opportunity to formalise the processes involved, including seeking general or exact solutions to optimisation problems. Indeed, in some (extreme)
cases, unlikely to be encountered by beginners, the technology may be insufficient for the task; Dubinsky (1995) showed a good example of this.

Again, the graphics calculator offers ways of linking concepts together, rather than merely producing numerical answers to optimisation questions. To illustrate this, the first screen in Figure 7 shows the same function as Figure 6, but with the value of the derivative showing at the same time. This sort of activity seems likely to help students see the connections between the value of the derivative and the turning point. The latter two screens show a check on the exact turning point at \( x = \frac{1}{\sqrt{3}} \).

![Figure 7: Calculator explorations associated with turning points](image)

In addition, other connections are made possible by the graphics calculator. Figure 8 shows another way of connecting the turning points with the derivative, by locating a root of the (numerical) derivative function in order to find a local maximum and using the calculator trace to check that there seems to be a relative minimum at \( x = \frac{1}{\sqrt{3}} \).

![Figure 8: Further calculator explorations related to optimisation](image)

The examples provided here for this function suggest several ways in which the experiences of both students and teachers may be altered, hopefully supported and enriched, through using graphics calculator features.

**Discontinuity**

Introductory students of calculus generally encounter continuous functions, although some elementary functions have points of discontinuity. By the (digital) nature of the technology, continuity is only approximated in graphical representations, both on calculators and on computers, for which screens are comprised of many discrete pixels. Despite this limitation, concepts of continuity and discontinuity can be developed with the assistance of technology. The rational function shown in Figure 9 is a good example.

![Figure 9: Representing a rational function that is discontinuous at \( x = 1 \)](image)

The 'hole' in the line on the screen suggests the idea of a removable discontinuity and the error shown in the table of values reinforces this. Successive zooming in, for either the graph or the table, allows students to experience the nature of the discontinuity, which is evident only at the exact value of \( x = 1 \), and not at points either side of this. Again, this sort of affordance can help build the concept, both for students exploring examples on their own calculators or a class exploring suitable examples with the support of a projected calculator.
Limits and asymptotic behaviour

As noted earlier, notions of limiting behaviour, while critical to understanding the calculus, can be dealt with informally at the early stages. Intuitions regarding the idea of a limit can be developed on a calculator by choosing smaller and smaller values of a variable (in the case of a limit as \(x \to 0\)) or larger and larger values of a variable (in the case of a limit as \(x \to \infty\)). Figure 10 shows the example of the limit, as \(x \to 0\) of \((\sin x)/x\).

![Figure 10: Numerical approximations to a limit as \(x \to 0\) of \((\sin x)/x\)](image)

Similarly, Figure 11, shows examples of the limit as \(x \to \infty\) of \((1 + 1/x)^x\), which is \(e\).

![Figure 11: Numerical approximations to a limit as \(x \to \infty\) of \((1 + 1/x)^x\)](image)

Graphical approximations of both limiting and asymptotic behaviour are also available and likely to help develop student insight, but space precludes discussing these further here.

Convergence of a series

Convergence is a problematic concept, as it involves notions of the infinite. A graphics calculator can offer some conceptual support, both graphically and numerically, by allowing an easy mechanism to generate and add many terms quickly. The example shown in Figure 12 converges very rapidly to \(e\), and is easily constructed and explored by students familiar with the graphics calculator.

![Figure 12: Exploring the convergence of the exponential series numerically and graphically](image)

Area under a curve

A major initial motivation for the study of integration is to find the area under a curve, which can be used to represent many important quantities in practice. A graphics calculator without CAS capabilities is able only to find good numerical approximations to such areas. An example is shown in Figure 13.

![Figure 13: Finding numerically the area above the x-axis under the curve given by \(y = 2 - x^2\)](image)
While the numerical approximations offered by the calculator are adequate for almost any practical purpose students might appreciate, the calculator does not provide much support for thinking about integration here beyond a visual image and the possibility of checking definite integrals obtained by standard means. As for the other examples provided in this section, the calculator is only one of the mechanisms for students to develop the ideas, and ought not be regarded as the only one.

**Differential equations**

A powerful way of thinking about elementary differential equations uses a slope-field diagram, such as that shown in Figure 14. While some more recent technologies (such as Casio’s *Classpad 300*) routinely provide a mechanism for drawing such a diagram, in this case two small calculator programs are needed. (Kissane & Kemp, 2006, pp 277-279) Together with earlier experience showing that many different functions can have the same derivative function, calculator explorations with examples like this will help students to appreciate that many possible functions can match the direction field displayed, depending on the initial conditions.

![Figure 14: Representing the differential equation dy/dx = x/2, with boundary condition (-4,2).](image)

**APPROPRIATE USE OF TECHNOLOGY**

The graphics calculator provides new affordances for teaching and learning calculus, but this is no guarantee that they will be used well by either students or teachers. When graphics calculators first became available, there were already significant discussions regarding the best ways to teach calculus and what to teach, especially in North America, where discussions of these kinds were referred to under the general heading of ‘calculus reform’. Eg, Cipra (1988, p. 1492) reported a concern that, despite promises that more time might be spent on the underlying ideas, in fact time was being spent on showing students which button to push. Similarly, Klein & Rosen (1997), responding to Mumford (1997) suggested that insufficient rigour and over-use of calculators were associated with the calculus reform movement. Such debate indicates that care needs to be taken to incorporate technology, including graphics calculators in particular, intelligently into the curriculum.

Similarly, in highlighting the ways in which teachers make use of technology in classrooms, Goos (2006) has made clear the many effects of contexts (including those in a school, particular to a person, related to an official curriculum or examination, and so on). While the graphics calculator offers new opportunities, teachers will need both support and help to make good use of these to support student learning. Part of the support will include textbooks, curricula and examination systems that recognise the possibilities and allow them to be used.

Both Solow (1994) and Stick (1997) have reported favourable results from using graphics calculators in college classrooms in the US, while Rochowicz (1997) reported a survey of college faculty that was mostly positive about such practices. It is clear from the field, from these and other researchers, that graphics calculators are no panacea, and require significant effort by teachers to use them well. The work of researchers like Guin and Trouche (1999) provides important elaborations of the process of ‘instrumental genesis’, rendering a calculator an efficient and effective tool for developing student understanding.
CONCLUSION

The graphics calculator affords both teachers and students a range of opportunities to encounter the calculus in a fresh way. In this paper, the major kinds of affordances associated with the calculator are identified and illustrated, and it is suggested that energy spent on taking advantage of these in classrooms will be worthwhile. Together, these seem to offer the best prospects for using technology to improve calculus teaching and learning on a wide scale in East Asia, although work will be needed to help and support teachers to do so effectively. The focus of this work should be on the development of the important concepts of the calculus.

REFERENCES


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