Palindromic Richness

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Outline

1. Rich Words: A Brief Overview
2. Properties & Examples
3. Recent Results
4. Further Work
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2 Properties & Examples

3 Recent Results

4 Further Work
What Are Rich Words?

- Vague Answer: finite and infinite words that are “rich” in palindromes in the utmost sense.
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- **Droubay-Justin-Pirillo, 2001:** any finite word $w$ of length $|w|$ contains at most $|w| + 1$ distinct palindromes (including the empty word $\varepsilon$).
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- **G.-Justin, 2007**: initiated a unified study of finite and infinite words that are characterized by containing the maximal number of distinct palindromes, called *rich words*. 
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- **G.-Justin, 2007:** initiated a unified study of finite and infinite words that are characterized by containing the maximal number of distinct palindromes, called *rich words*.

- **Ambrož-Frougny-Masáková-Pelantová, 2005:** independent work on “full words”, following earlier work of Brlek-Hamel-Nivat-Reutenauer, 2004.
Definition

A finite word $w$ is *rich* iff $w$ contains exactly $|w| + 1$ distinct palindromes.
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An infinite word is **rich** iff all of its factors are rich.
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A finite word $w$ is \textit{rich} iff $w$ contains exactly $|w| + 1$ distinct palindromes.

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- $abac$ is rich, whereas $abca$ is \textbf{not} rich.
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- $a^\omega = aaaaaa \cdots$ and $ab^\omega = abbb \cdots$ are rich.
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- \( abac \) is rich, whereas \( abca \) is not rich.
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- \( a^\omega = aaaaaa \cdots \) and \( ab^\omega = abbb \cdots \) are rich.
- \( (ab)^\omega = abababab \cdots \) and \( (aba)^\omega = ababaaba \cdots \) are rich.
- \( abc \) is rich, but \( (abc)^\omega = abcabcabc \cdots \) is not rich.
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Characteristic Properties

Characteristic Property 1 (Droubay-Justin-Pirillo, 2001)

A finite word $w$ is rich if and only if every prefix (resp. suffix) of $w$ has a unioccurrent palindromic suffix (resp. prefix).
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To see this . . .

- Let $P(w)$ denote the number of palindromic factors of $w$. 
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- Let $P(w)$ denote the number of palindromic factors of $w$.
- For any word $u$ and letter $x$,

$$P(ux) = \begin{cases} P(u) & \text{if } ux \text{ does not have a unioccurrent palindromic suffix}, \\ P(u) + 1 & \text{if } ux \text{ has a unioccurrent palindromic suffix}. \end{cases}$$
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- Therefore, by induction, $P(w)$ is the number of prefixes of $w$ that have a unioccurrent palindromic suffix.
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- Therefore, by induction, $P(w)$ is the number of prefixes of $w$ that have a unioccurrent palindromic suffix.
- Hence $P(w) \leq |w| + 1$. 

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- Therefore, by induction, $P(w)$ is the number of prefixes of $w$ that have a unioccurrent palindromic suffix.
- Hence $P(w) \leq |w| + 1$.
- In particular $P(w) = |w| + 1$ (i.e., $w$ is rich) if and only if each prefix of $w$ has a unioccurrent palindromic suffix.
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A new palindrome is introduced at each position in a rich word.

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Infinite case of Characteristic Property 1:

- An infinite word \( w \) is rich if and only if every prefix of \( w \) has a unioccurrent palindromic suffix.

A new palindrome is introduced at each position in a rich word.

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Characteristic Properties

- Let $u$ be a factor of a finite or infinite word $w$.
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A finite or infinite word $w$ is rich if and only if for each palindromic factor $p$ of $w$, every *complete return* to $p$ in $w$ is a palindrome.
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- ($\Rightarrow$): Suppose $w$ is rich, but contains a non-palindromic complete return $r$ to a palindrome $p$. 
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**Proof:**

1. $(\Rightarrow)$: Suppose $w$ is rich, but contains a non-palindromic complete return $r$ to a palindrome $p$.
2. Then $r = pup$ for some non-palindromic word $u$. 
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- (\( \Rightarrow \)): Suppose \( w \) is rich, but contains a non-palindromic complete return \( r \) to a palindrome \( p \).
- Then \( r = pup \) for some non-palindromic word \( u \).
- But then \( r \) does not have a unioccurrent palindromic suffix, a contradiction.
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- $(\Leftarrow)$: Suppose not. Let $u$ be a factor of $w$ of minimal length such that $u$ is not rich.
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- $(\Leftarrow)$: Suppose not. Let $u$ be a factor of $w$ of minimal length such that $u$ is not rich.
- Then $u = xvy$ with $x, y$ letters. By minimality $xv$ is rich, and the longest palindromic suffix $p$ of $u$ occurs more than once in $u$. 
Characteristics of Properties

Let $u$ be a factor of a finite or infinite word $w$.

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A finite or infinite word $w$ is rich if and only if for each palindromic factor $p$ of $w$, every complete return to $p$ in $w$ is a palindrome.

Proof:

$(\Leftarrow)$: Suppose not. Let $u$ be a factor of $w$ of minimal length such that $u$ is not rich.

Then $u = xvy$ with $x, y$ letters. By minimality $xv$ is rich, and the longest palindromic suffix $p$ of $u$ occurs more than once in $u$.

Since all complete returns to palindromes are palindromes, we reach a contradiction to the maximality of $p$. 
Characteristic Properties

Let $\tilde{v}$ denote the *reversal* of a word $v$. Example: $v = abc$, $\tilde{v} = cba$.

**Characteristic Property 4 (Bucci-De Luca-G.-Zamboni, 2008)**

For any finite or infinite word $w$, the following conditions are equivalent:

i) $w$ is rich;

ii) for each factor $v$ of $w$, every factor of $w$ beginning with $v$ and ending with $\tilde{v}$ and containing no other occurrences of $v$ or $\tilde{v}$ is a palindrome.
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**Proof:**

i) $\Rightarrow$ ii): Let $u = v \cdots \tilde{v}$. If $v$ is a palindrome, then $u$ is a palindrome.
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**Proof:**

i) $\Rightarrow$ ii): Let $u = v \cdots \tilde{v}$. If $v$ is a palindrome, then $u$ is a palindrome. Otherwise, for non-palindromic $v$, suppose $u$ is not a palindrome ...
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\[ u = \begin{array}{c|c|c} v & \cdots & \tilde{v} \\ \hline p & \cdots & p \end{array} \]

($p$ is longest palindromic suffix of $u$)

**complete return to** $p \rightarrow \text{palindrome}$
Let $\tilde{\nu}$ denote the reversal of a word $\nu$. Example: $\nu = abc$, $\tilde{\nu} = cba$.

**Characteristic Property 4 (Bucci-De Luca-G.-Zamboni, 2008)**

For any finite or infinite word $w$, the following conditions are equivalent:

1. $w$ is rich;
2. for each factor $\nu$ of $w$, every factor of $w$ beginning with $\nu$ and ending with $\tilde{\nu}$ and containing no other occurrences of $\nu$ or $\tilde{\nu}$ is a palindrome.

**Proof:**

1. $i) \Rightarrow ii)$: Let $u = \nu \cdots \tilde{\nu}$. If $\nu$ is a palindrome, then $u$ is a palindrome. Otherwise, for non-palindromic $\nu$, suppose $u$ is **not** a palindrome . . .

$$u = \begin{array}{c|c|c}
\nu & \cdots & \tilde{\nu} \\
p & \cdots & p \\
\end{array}$$

($p$ is longest palindromic suffix of $u$) . . . contradiction!

complete return to $p \rightarrow$ **palindrome**
Characteristic Properties

Let \( \tilde{v} \) denote the *reversal* of a word \( v \). Example: \( v = abc, \tilde{v} = cba \).

**Characteristic Property 4 (Bucci-De Luca-G.-Zamboni, 2008)**

For any finite or infinite word \( w \), the following conditions are equivalent:

i) \( w \) is rich;

ii) for each factor \( v \) of \( w \), every factor of \( w \) beginning with \( v \) and ending with \( \tilde{v} \) and containing no other occurrences of \( v \) or \( \tilde{v} \) is a palindrome.

**Proof:**

Conversely, ii) \( \Rightarrow \) every complete return to a palindromic factor \( v \) (\( = \tilde{v} \)) is a palindrome.
Let $\tilde{v}$ denote the *reversal* of a word $v$. Example: $v = abc$, $\tilde{v} = cba$.

**Characteristic Property 4** (Bucci-De Luca-G.-Zamboni, 2008)

For any finite or infinite word $w$, the following conditions are equivalent:

i) $w$ is rich;

ii) for each factor $v$ of $w$, every factor of $w$ beginning with $v$ and ending with $\tilde{v}$ and containing no other occurrences of $v$ or $\tilde{v}$ is a palindrome.

**Proof:**

- Conversely, ii) $\Rightarrow$ every complete return to a palindromic factor $v$ ($= \tilde{v}$) is a palindrome.

- Thus $w$ is rich by Characteristic Property 3.
Rich Examples

**Purely Periodic Rich Infinite Words**

\[(abcba)^\omega = abcbaabcbaabcba \cdots\]
Rich Examples

Purely Periodic Rich Infinite Words

\( (abcba) \omega = abcbaabcbaabcba \cdots \)

\( (aab^k aabab) \omega = aab^k aabab aab^k aabab aabab \cdots \) with \( k \geq 0 \)
Rich Examples

Purely Periodic Rich Infinite Words

- \((abcba)^\omega = abcbaabcbaabcba \cdots\)
- \((aab^k aabab)^\omega = aab^k aababaab^k aababaab^k aabab \cdots\) with \(k \geq 0\)
- \(v^\omega = vvv \cdots\) is rich \(\iff v^2\) is rich
Rich Examples

Purely Periodic Rich Infinite Words

- \((abcba)^\omega = abcbaabcbaabcba \cdots\)
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**Other Rich Infinite Words**
- \(abcd\omega = abcd\omega \cdots\)
Rich Examples

Purely Periodic Rich Infinite Words

- \((abcba)^\omega = abcbaabcbaabcba \cdots\)
- \((aab^k aabab)^\omega = aab^k aababaab^k aababaab^k aabab \cdots\) with \(k \geq 0\)
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Other Rich Infinite Words

- \(abcd^\omega = abcddd \cdots\)
- \(aba^2 ba^3 ba^4 ba^5 b \cdots\)
Rich Examples

Purely Periodic Rich Infinite Words

1. \((abcba)^\omega = abcbaabcbaabcba \cdots\)
2. \((aab^k aabab)^\omega = aab^k aababaab^k aababaab^k aabab \cdots\) with \(k \geq 0\)
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Other Rich Infinite Words

1. \(abcd^\omega = abcddd \cdots\)
2. \(aba^2 ba^3 ba^4 ba^5 b \cdots\)
3. \(\lim_{n \to \infty} \sigma^n(a) = ababbabbbbbababbababbbbbbabbaba \cdots\)
   where \(\sigma: a \mapsto aba, b \mapsto bb\) (Cassaigne, 1997).
Rich Examples

Purely Periodic Rich Infinite Words

- \((abcba)\omega = abcbaabcbaabcba \cdots\)
- \((aab^k aabab)\omega = aab^k aababaab^k aababaab^k aabab \cdots\) with \(k \geq 0\)
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Other Rich Infinite Words

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- Fibonacci word: \(f = \lim_{n \to \infty} \varphi^n(a) = abaababaabaababaaba \cdots\)
  where \(\varphi : a \mapsto ab, b \mapsto a\).
Rich Examples

Purely Periodic Rich Infinite Words

- \((abcba)\omega = abcbaabcbaabcba\ldots\)
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Other Rich Infinite Words

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- **Fibonacci word:** \(f = \lim_{n \to \infty} \varphi^n(a) = abaababaabaababaaba\ldots\) where \(\varphi: a \mapsto ab, b \mapsto a\).
- **Tribonacci word:** \(r = \lim_{n \to \infty} \theta^n(a) = abacabaabacabacabacabaaba\ldots\) where \(\theta: a \mapsto ab, b \mapsto ac, c \mapsto a\).
Rich Examples

Purely Periodic Rich Infinite Words
- \((abcba)\omega = abcbaabcbaabcba \cdots\)
- \((aab^k aabab)\omega = aab^k aababaab^k aababaab^k aabab \cdots\) with \(k \geq 0\)
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Other Rich Infinite Words
- \(abcd\omega = abcdd \cdots\)
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  where \(\theta : a \mapsto ab, b \mapsto ac, c \mapsto a\).
- \(\psi_k(f)\) where \(\psi_k : a \mapsto aab^k aabab, b \mapsto bab, k \geq 0\).
Rich words have appeared in many different contexts; they include:

- **Sturmian and episturmian words**
  Droubay-Justin-Pirillo, 2001: characteristic property 1
  Anne-Zamboni-Zorca, 2005: characteristic property 3
  Bucci-De Luca-G.-Zamboni, 2008: characterization of recurrent rich infinite words

- **Complementation-symmetric Rote sequences**

- **Symbolic codings of trajectories of symmetric interval exchange transformations** – Ferencezi-Zamboni, 2008

- **A certain class of words associated with $\beta$-expansions where $\beta$ is a simple Parry number**

- **Infinite words with “abundant palindromic prefixes”**
  Introduced by Fischler in 2006 in relation to Diophantine approximation
If a finite word $w$ is rich, then its *reversal* $\tilde{w}$ is also rich.

Example: $w = aabac$ and $\tilde{w} = cabaa$ are both rich.
Basic Properties & Results (G.-Justin, 2007)

- If a finite word $w$ is rich, then its *reversal* $\tilde{w}$ is also rich.
  
  **Example:** $w = aabac$ and $\tilde{w} = cabaa$ are both rich.

- If $w$ and $w'$ are rich with the same set of palindromic factors, then they are *abelianly equivalent*, i.e., $|w|_x = |w'|_x$ for all letters $x$. 
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Palindromic closure preserves richness.
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- Palindromic closure preserves richness.
  The palindromic closure of a word $v$, denoted by $v^+$, is the unique shortest palindrome beginning with $v$. 
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Examples:
$(race)^+ =$
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**Palindromic closure preserves richness.**

The *palindromic closure* of a word $v$, denoted by $v^+$, is the unique shortest palindrome beginning with $v$.

**Examples:**
$(race)^+ = race$
Basic Properties & Results (G.-Justin, 2007)

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  The \textit{palindromic closure} of a word $v$, denoted by $v^+$, is the unique shortest palindrome beginning with $v$.

  Examples:
  
  $(race)^+ = race car$
Basic Properties & Results (G.-Justin, 2007)

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  The *palindromic closure* of a word $v$, denoted by $v^+$, is the unique shortest palindrome beginning with $v$.

  Examples:
  - $(race)^+ = race \text{ car}$
  - $(tops)^+ =$
If a finite word $w$ is rich, then its reversal $\tilde{w}$ is also rich.

Example: $w = aabac$ and $\tilde{w} = cabaa$ are both rich.

If $w$ and $w'$ are rich with the same set of palindromic factors, then they are **abelianly equivalent**, i.e., $|w|_x = |w'|_x$ for all letters $x$.

Palindromic closure preserves richness.

The **palindromic closure** of a word $\nu$, denoted by $\nu^+$, is the unique shortest palindrome beginning with $\nu$.

Examples:

$(race)^+ = race\ car$

$(tops)^+ = tops$
Basic Properties & Results (G.-Justin, 2007)

- If a finite word $w$ is rich, then its reversal $\tilde{w}$ is also rich.
  Example: $w = aabac$ and $\tilde{w} = cabaa$ are both rich.

- If $w$ and $w'$ are rich with the same set of palindromic factors, then they are *abelianly equivalent*, i.e., $|w|_x = |w'|_x$ for all letters $x$.

- Palindromic closure preserves richness.

  The *palindromic closure* of a word $v$, denoted by $v^+$, is the unique shortest palindrome beginning with $v$.

  Examples:
  $(race)^+ = race car$
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Basic Properties & Results (G.-Justin, 2007)

- If a finite word \( w \) is rich, then its \emph{reversal} \( \tilde{w} \) is also rich.  
  \textbf{Example:} \( w = aabac \) and \( \tilde{w} = cabaa \) are both rich.

- If \( w \) and \( w' \) are rich with the same set of palindromic factors, then they are \emph{abelianly equivalent}, i.e., \( |w|_x = |w'|_x \) for all letters \( x \).

- Palindromic closure preserves richness.
  The \emph{palindromic closure} of a word \( v \), denoted by \( v^+ \), is the unique shortest palindrome beginning with \( v \).
  \textbf{Examples:}  
  \((race)^+ = race car \)  
  \((tops)^+ = top spot \)  
  \((party)^+ = \)
Basic Properties & Results (G.-Justin, 2007)

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  The \textit{palindromic closure} of a word $v$, denoted by $v^+$, is the unique shortest palindrome beginning with $v$.

Examples:
- $(race)^+ = race car$
- $(tops)^+ = top spot$
- $(party)^+ = party trap$
- $(tie)^+ =$
Basic Properties & Results (G.-Justin, 2007)

- If a finite word $w$ is rich, then its reversal $\tilde{w}$ is also rich.
  
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  Examples:
  
  $(race)^+ = race car$
  
  $(tops)^+ = top spot$
  
  $(party)^+ = party trap$
  
  $(tie)^+ = tie$
If a finite word $w$ is rich, then its reversal $\tilde{w}$ is also rich.

Example: $w = aabac$ and $\tilde{w} = cabaa$ are both rich.

If $w$ and $w'$ are rich with the same set of palindromic factors, then they are abelianly equivalent, i.e., $|w_x| = |w'_x|$ for all letters $x$.

Palindromic closure preserves richness.

The palindromic closure of a word $v$, denoted by $v^+$, is the unique shortest palindrome beginning with $v$.

Examples:
$(race)^+ = race car$
$(tops)^+ = top spot$
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Basic Properties & Results (G.-Justin, 2007)

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  The palindromic closure of a word $v$, denoted by $v^+$, is the unique shortest palindrome beginning with $v$.

  Examples:
  
  $(race)^+ = race car$
  $(tops)^+ = top spot$
  $(party)^+ = party trap$
  $(tie)^+ = tie it$
  $(abac)^+ =$
Basic Properties & Results (G.-Justin, 2007)

- If a finite word $w$ is rich, then its reversal $\bar{w}$ is also rich.
  Example: $w = aabac$ and $\bar{w} = cabaa$ are both rich.

- If $w$ and $w'$ are rich with the same set of palindromic factors, then they are *abelianly equivalent*, i.e., $|w|_x = |w'|_x$ for all letters $x$.

- Palindromic closure preserves richness.
  The *palindromic closure* of a word $v$, denoted by $v^+$, is the unique shortest palindrome beginning with $v$.
  
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  $(race)^+ = race car$
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  $(abac)^+ = abac$
If a finite word $w$ is rich, then its *reversal* $\tilde{w}$ is also rich.

Example: $w = aabac$ and $\tilde{w} = cabaa$ are both rich.

If $w$ and $w'$ are rich with the same set of palindromic factors, then they are *abelianly equivalent*, i.e., $|w|_x = |w'|_x$ for all letters $x$.

Palindromic closure preserves richness.

The *palindromic closure* of a word $v$, denoted by $v^+$, is the unique shortest palindrome beginning with $v$.

Examples:

$(race)^+ = race car$
$(tops)^+ = top spot$
$(party)^+ = party trap$
$(tie)^+ = tie it$
$(abac)^+ = abac aba$
Sturmian Words Are Rich

**Theorem (de Luca, 1997)**

An infinite word \( s \) over \( \{a, b\} \) is a **standard Sturmian word** if and only if there exists an infinite word \( \Delta = x_1x_2x_3 \cdots \) over \( \{a, b\} \) (not of the form \( ua^\omega \) or \( ub^\omega \)), called the **directive word** of \( s \), such that

\[
s = \lim_{n \to \infty} Pal(x_1x_2 \cdots x_n).
\]
Sturmian Words Are Rich

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s = \lim_{n \to \infty} \text{Pal}(x_1x_2 \cdots x_n).
\]

\( \text{Pal} \) is the **iterated palindromic closure** function:

\[
\text{Pal}(\varepsilon) = \varepsilon \text{ (empty word)} \quad \text{and} \quad \text{Pal}(wx) = (\text{Pal}(w)x)^+
\]

for any word \( w \) and letter \( x \). **Example:** \( \text{Pal}(aba) = \)
Sturmian Words Are Rich

**Theorem (de Luca, 1997)**

An infinite word $s$ over $\{a, b\}$ is a **standard Sturmian word** if and only if there exists an infinite word $\Delta = x_1 x_2 x_3 \cdots$ over $\{a, b\}$ (not of the form $ua^\omega$ or $ub^\omega$), called the **directive word** of $s$, such that

$$s = \lim_{n \to \infty} \text{Pal}(x_1 x_2 \cdots x_n).$$

$\text{Pal}$ is the **iterated palindromic closure** function:

$$\text{Pal}(\varepsilon) = \varepsilon \text{ (empty word)} \quad \text{and} \quad \text{Pal}(wx) = (\text{Pal}(w)x)^+$$

for any word $w$ and letter $x$. **Example:** $\text{Pal}(aba) = a$
Properties & Examples

Sturmian Words Are Rich

**Theorem** (de Luca, 1997)

An infinite word $s$ over $\{a, b\}$ is a **standard Sturmian word** if and only if there exists an infinite word $\Delta = x_1 x_2 x_3 \cdots$ over $\{a, b\}$ (not of the form $ua^\omega$ or $ub^\omega$), called the **directive word** of $s$, such that

$$s = \lim_{n \to \infty} Pal(x_1 x_2 \cdots x_n).$$

- $Pal$ is the **iterated palindromic closure** function:

  $$Pal(\varepsilon) = \varepsilon \text{ (empty word)} \quad \text{and} \quad Pal(wx) = (Pal(w)x)^+$$

  for any word $w$ and letter $x$. **Example:** $Pal(aba) = ab$
Sturmian Words Are Rich

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Properties & Examples

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- The *Fibonacci word* is directed by $\Delta = (ab)^\omega = ababab \cdots$.

  That is: $f = \text{Pal}(ababab \cdots) = \underline{a}b\underline{a}ba\underline{b}aba \underline{b}aba \cdots$. 
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- Palindromic closure preserves richness $\Rightarrow$ $Pal$ does too $\Rightarrow$ Sturmian words are RICH.
Episturmian Words Are Rich Too

\{a, b\} \rightarrow \mathcal{A} \text{ (finite alphabet) gives standard episturmian words.}

**Theorem (Droubay-Justin-Pirillo, 2001)**

An infinite word \( s \) over \( \mathcal{A} \) is a standard episturmian word if and only if there exists an infinite word \( \Delta = x_1x_2x_3 \cdots \) over \( \mathcal{A} \) such that

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Episturmián Words Are Rich Too

\{a, b\} \longrightarrow \mathcal{A} \text{ (finite alphabet) gives } \textit{standard episturmián words}.

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\[
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\]

**Example**

\(\Delta = (abc)\omega = abcabcabc \cdots\) directs the \textit{Tribonacci word}:

\[
r = \underline{abacaba}abacaba \underline{abacaba}abacabaabacabaca \underline{aba}baca \cdots
\]
Outline

1. Rich Words: A Brief Overview
2. Properties & Examples
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Let $w$ be a finite or infinite word.
A Connection Between Palindromic & Factor Complexity

- Let $w$ be a finite or infinite word.
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Allouche-Baake-Cassaigne-Damanik, 2003: for any aperiodic infinite word $w$,

$$P_w(n) \leq \frac{16}{n} C_w \left( n + \left\lfloor \frac{n}{4} \right\rfloor \right) \quad \text{for all } n \in \mathbb{N}.$$
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$$P_w(n) + P_w(n + 1) \leq C_w(n + 1) - C_w(n) + 2 \quad \text{for all } n \in \mathbb{N} \quad (*)$$
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\]

**Bucci-De Luca-G.-Zamboni, 2008:** infinite words \( w \) for which

\[
P_w(n) + P_w(n + 1) \text{ reaches the upper bound in } (\star) \text{ for every } n \text{ are rich .}
\]
A Connection Between Palindromic & Factor Complexity

Theorem A (Bucci-De Luca-G.-Zamboni, 2008)

For any infinite word $w$ with set of factors $F(w)$ closed under reversal, the following conditions are equivalent:

(I) all complete returns to any palindrome in $w$ are palindromes;

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Complementation-symmetric Rote sequences:

- Infinite words over $\{a, b\}$ with factors closed under both complementation and reversal, and such that $\mathcal{C}(n) = 2n$ for all $n$. 

Recent Results

A Connection Between Palindromic & Factor Complexity

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Sturmian words:

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Finite Case of Theorem A

Using completely different methods . . .

**Theorem** (de Luca-G.-Zamboni, 2008)

For any finite word $w$, the following two conditions are equivalent:

i) $w$ is a rich palindrome;

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Every finite Sturmian word is trapezoidal, but not conversely. E.g., \( aabb \) is trapezoidal, but not Sturmian.
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- Every trapezoidal word is rich, but not conversely. E.g., $aabbaa$. 
Outline

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Amy Glen (LaCIM)
More Stuff on Rich Words


- *almost rich words*: only a finite number of prefixes do not have a unioccurent palindromic suffix
More Stuff on Rich Words

G.-Justin-Widmer-Zamboni, *Palindromic richness, 2008*

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Example: \((pq)^\omega = pqpqpq \cdots\) where \(p, q\) are palindromes
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Open Problems

- Characterize substitutions that preserve (almost) richness
- Enumeration of rich words
Thank You!

Dammit, I’m mad!

U R 2, R U?

* Both phrases are rich palindromes! *