The graphics calculator as an investigative tool

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Introduction

The graphics calculator seems to be frequently misunderstood, especially by those who are unaccustomed to using it. (Kissane, 1997a) For example, it is sometimes regarded as a device to assist students to pass examinations, perhaps understandably in places where its use has become widespread because of changes to examination rules. Similarly, it is also interpreted sometimes as a device whose main significance is related to drawing graphs of functions, again not surprisingly in situations where it is described as a 'graphing' calculator. As a further example, it is sometimes dismissed as a form of technology because it is not commonly used by professional people such as industrial mathematicians, statisticians, scientists and engineers, most of whom routinely use desktop or laptop computers for mathematical work.

In contrast to these opinions, I would argue that the significance of the graphics calculator for education is that it offers opportunities for investigation that would not otherwise be available to students. That is, its significance is for learning rather than the assessment of learning. This includes aspects of graphing functions, but many other aspects of mathematics as well. Although less powerful than other sorts of computers, it is of educational importance because it is potentially accessible to all students, rather than to only the privileged few. (Kissane 1995a,1996)

In support of these claims, this paper outlines very briefly three of the ways in which a graphics calculator can be regarded as an investigative device, described as the most important metaphor for the device by Kissane (1995b). For convenience, the Casio cfx-9850GB Plus graphics calculator is used to illustrate the possibilities.

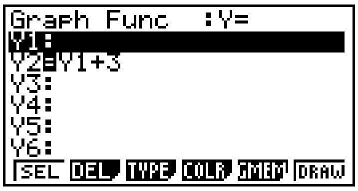
Experimental mathematics

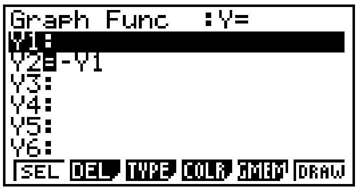
A graphics calculator can provide students with opportunities to experiment with mathematical ideas and objects, a powerful approach to learning. One aspect of this is that it puts the learner in control to an extent, making decisions about what to do next and interpreting the results. Perhaps the most obvious example of this is in the area of data analysis, superbly handled by modern graphics calculators. The idea of *exploratory data analysis*, a product of the computer age, becomes accessible when the data are stored in a calculator: data can be checked, edited, transformed, augmented, analysed and re-analysed. Extensive examples of this are given in Kissane (1997).

Function transformer

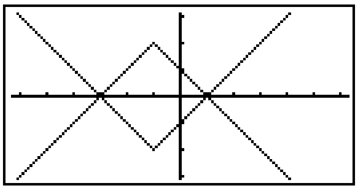
Understanding functions is aided by a calculator providing access to the 'rule of three': symbolic, graphical and numerical representations are all available for a function and can be easily experimented with by students. A *function transformer* (Kissane, 1997, p.47) is a particular way of experimenting, in which one function is transformed to another, allowing students to see the effects of particular transformations (such as taking the absolute value, adding a constant, multiplicative scaling, finding the opposite, etc.). Such transformations can be regarded as 'building blocks' to more complicated functions.

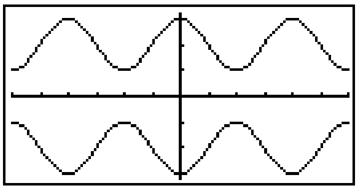
The two screens below (from Kissane 1997, p.48) show a pair of function transformers, the first of which 'adds 3' to the function definition in Y1, while the second 'finds the opposite' of the function defined in Y1. For students familiar with the calculator, these are easy to define and easy to explore both graphically and numerically.





Here are two examples of the opposite transformer in action:



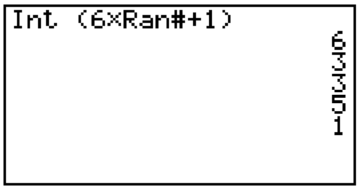


Experimenting with transformations and studying their effects is potentially useful experience for students coming to grips with more complicated functions than the elementary ones. For older students, a particularly powerful transformer is the derivative transformer, for which a function is defined as the derivative of another function.

Randomisation

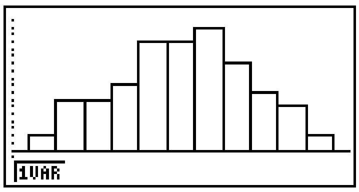
All graphics calculators contain (psuedo-) random number generators, which can be manipulated by students to engage in random experiments of various kinds. At a relatively unsophisticated level students can generate random data in scientific calculator mode by repeatedly pressing the EXE key.

For example, the screen at the right shows a calculator simulating dice tosses using a transformation of the Ran# command, which generates random numbers uniformly on (0,1).



In a more sophisticated way, a larger set of simulated data can be automatically generated and stored on the calculator and then electronically transferred to the statistics part of the calculator for analysis.

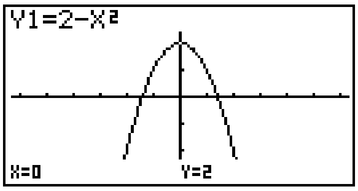
For example, the screen at right shows a histogram of 255 simulated tosses of a pair of dice. (Boys, Harradine & Kissane 1999)

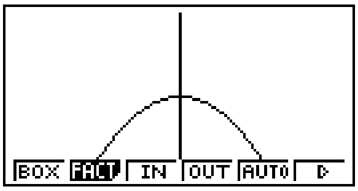


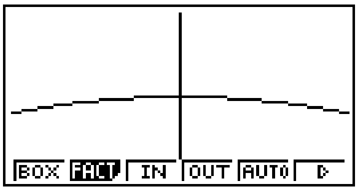
The advantage of experimenting with randomness on the calculator is that, by its very nature, each attempt will produce different results; however, careful attention to similarities will help students see the very important 'big pictures'. An extensive example of using a graphics calculator to learn about randomness ('Dealing with randomness') is available from the Student-Teacher resources page at http://www.sharp-au.com.au/menu/calcframe.htm.

Local linearity

The idea of a derivative is critical to the introductory study of calculus. However, formal symbolic definitions are not always helpful to students, who are required firstly to come to terms with the (difficult) idea of a limit. With a graphics calculator, the idea of a derivative of a function can be addressed directly, by considering the gradient of the curve itself (rather than tangents or secants to it), the central aspect of which is local linearity. Put simply, if you zoom in far enough on the graph of any continuous function, the curve will become 'straight'. The derivative of the function at any point is just the gradient of the curve at that point.







A calculator allows for this idea to be readily explored, as the three screens above suggest. By repeatedly zooming in, the idea that the gradient of the curve is zero at the apex of the parabola is relatively easy for students to come to terms with.

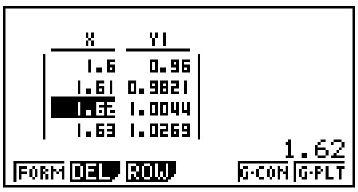
Multiple perspectives

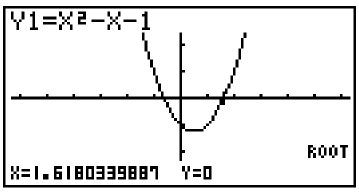
The 'rule of three' is one example of the capacity of a calculator to aid investigation by providing multiple perspectives on the same idea (that of a function). However, it is certainly not the only example. A graphics calculator frequently provides opportunities for students to investigate mathematical ideas from more than one angle. Such investigation is likely to lead to richer conceptual structures and thus provides a powerful new way of learning.

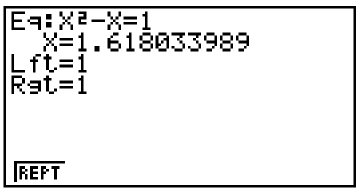
Solving an equation

An extended illustration of this idea in the context of solving an equation is given in Kissane (1996). A graphics calculator can be used to investigate solutions to an equation numerically (by looking at tabulated values, for example), graphically (by zooming in on graphs of appropriate functions, manually or automatically) and even directly (using an iterative numerical equation solver).

The screens below show an example of this for the quadratic equation that gives rise to the golden ratio, $x^2 - x = 1$:

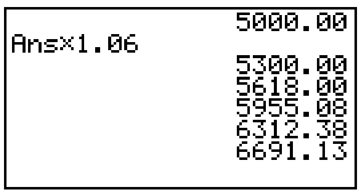


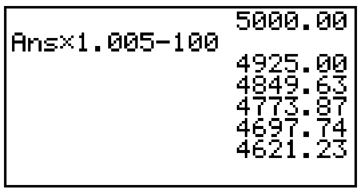




Compound interest

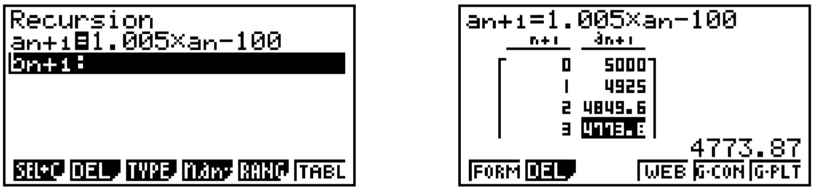
Compound interest is an important, but difficult, topic for students, as it provides access to understanding their financial affairs in adulthood, particularly large purchases such as houses and cars. The idea can be explored in several ways on a calculator. The first screen at the right shows how repeated presses of the EXE key show the accumulated amounts for $5000 borrowed at 6% interest compounded annually. Each key press shows the amount after a year. The second screen shows how the calculator can be used similarly to deal with the (more realistic, but much more



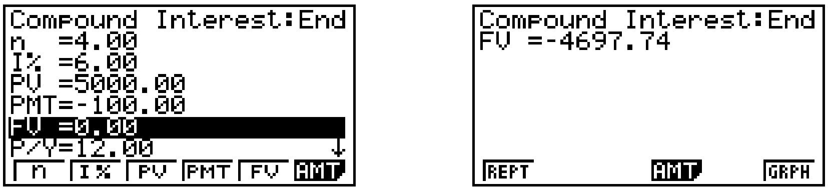


complicated) case of borrowing $5000 at 6% compound interest, but with monthly repayments of $100. Each key press now shows the amount owing at the end of each month.

The same ideas can be dealt with recursively in the calculator as shown below, making the recursive relationships explicit and thus more transparent.



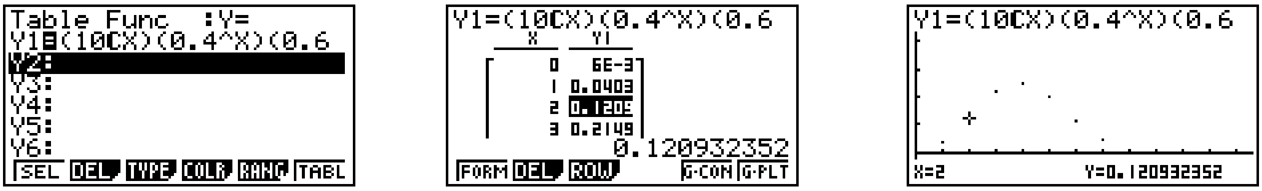
As a third alternative, the calculator can deal with financial mathematics directly, as suggested below.



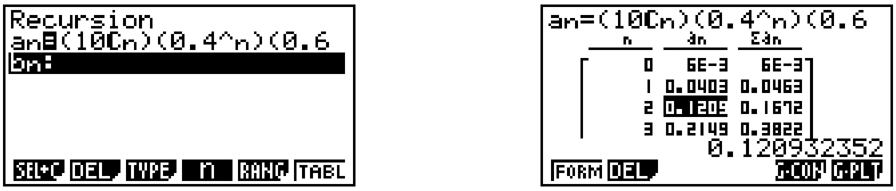
A calculator allows students to investigate readily similarities and differences between these ways of thinking about and calculating compound interest.

Probability distributions

As a final example, students can explore probability distributions from various perspectives and in various ways. The binomial distribution can be generated by making a table of values of the relevant distribution function. The example below shows the case of tossing ten times a biased coin, for which the probability of a head is 0.4. Once tabulated, the probabilities can be plotted as well.



Alternatively, the recursive area of the calculator can be used to generate similar values, with the additional advantage that they can be automatically accumulated to give the cumulative binomial distribution, as shown below.



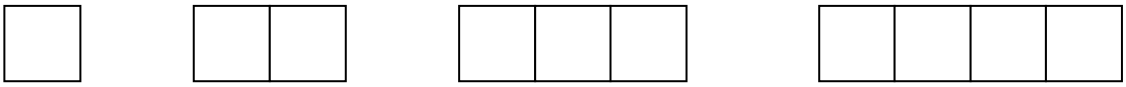
The calculator also offers access to the binomial distribution directly in the Statistics area.

Making links

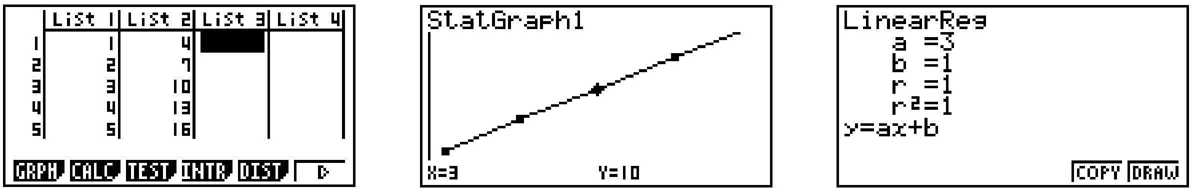
A theme of each of the previous two sections is that a calculator provides opportunities for students to make links between mathematical ideas. Some of these links cross further boundaries than others, however. Thus, the links between a function and its graphs are quite close, while those between data analysis and algebra are a little more remote. A student with access to a graphics calculator has more chance to investigate links near and far. Three examples of links between statistics and other aspects of mathematics are given below.

Functions and statistics

An important aspect of functions concerns their representation symbolically (as well as numerically and graphically). When students investigate practical situations to produce tables of values such as those below, they can often see the horizontal patterns but find the vertical patterns more elusive. This particular example shows the number of matchsticks needed to build successive rectangular patterns of the kind shown below.



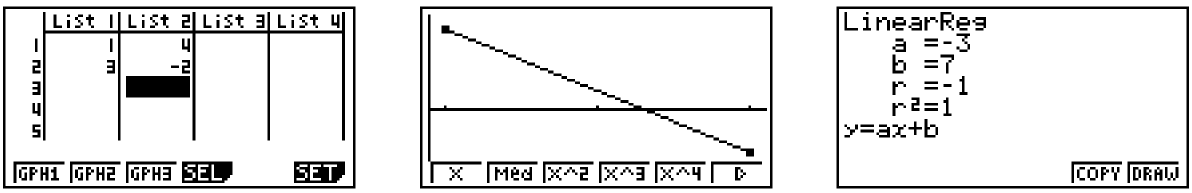
The data can be entered into the calculator as ordered pairs and the regression aspects of the calculator used to see which function appears to fit them.



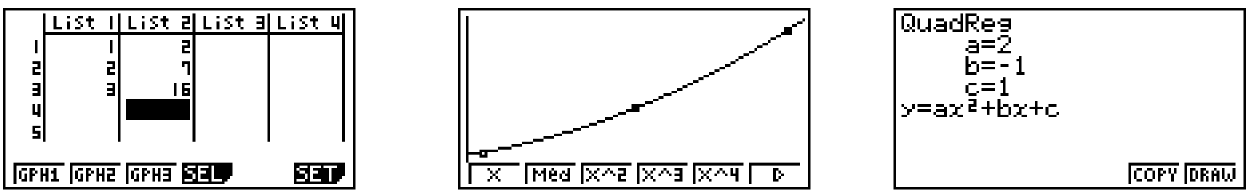
In this case, the function is linear, $f(n) = 3n + 1$.

Geometry and statistics

A similar process to that described above can be used to find a line through two points. This example shows that the line joining the points (1,4) and (3,-2) can be represented by $y = -3x + 7$.



On a more sophisticated note, the (only) quadratic function that passes through the points (1,2), (3,7) and (3,16) is readily seen to be $y = 2x^2 – x + 1$, providing students choose a quadratic model to suit the situation, analogously to choosing a linear model for the case of two points.

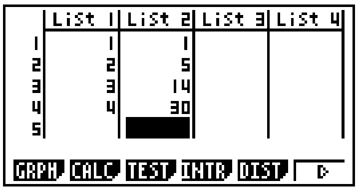


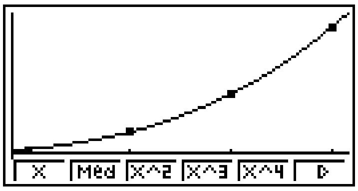
Note that it is not necessary for students to understand the statistical and distributional concepts associated with regression to use their calculator to investigate links of these kinds; it is perfectly adequate to regard the calculator as a 'curve-fitting' device. Arguably, exploring links like these is of value both for coordinate geometry and for statistics.

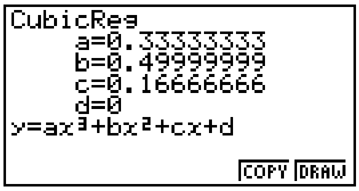
Series and statistics

The young Gauss and (with some help) our own students can see the formula involved in finding the sum of successive positive integers. However, it is rather more difficult to do the same for the sums of successive squares. The result can be induced from the raw data, by using the statistics part of the calculator. In this case the raw data arise from investigating by hand successive terms of the series:

$1^2 = 1$
$1^2 + 2^2 = 5$
$1^2 + 2^2 + 3^2 = 14$
$1^2 + 2^2 + 3^2 + 4^2 = 30$ … and so on.







The calculator suggests that the general result is

$$f(n) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \ ,$$

assuming that the numerical approximations represent the fractions shown. This result depends on students choosing to fit a cubic model to the data (reasoning inductively from the linear series, for which the sum is a quadratic).

This result can of course be simplified and factorised (by hand) to the result

$$f(n) = \frac{n}{6}(2n^2 + 3n + 1) = \frac{n(n+1)(2n+1)}{6} \ .$$

This result can be verified against the original (or some new) data.

Conclusion

This paper has suggested that the graphics calculator has very considerable potential for student learning because of its inherent capabilities to foster students investigation. The essence of investigation is student control, which is rendered more possible with the accessible and affordable technology of the graphics calculator. Three aspects of this have been elaborated through the medium of a few examples: experimental mathematics, multiple perspectives and making links. Space has prevented a more complete treatment of investigation or even adequate elaboration of the examples shown, which have been chosen as representative of many possibilities. To focus on the graphics calculator as an examination tool is to grossly misunderstand its importance to mathematics education.

References

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Invited reader reactions …

1. Readers are invited to react to the central argument of this paper that the main significance of the graphics calculator is for investigative and exploratory student work, rather than for assessment purposes. (Incidentally, it is assumed in this stance that mathematical investigation is essentially a learning activity rather than itself being an assessment activity, although I appreciate that there are some settings where the latter view predominates. When I use the word 'investigation', I am *not* referring to a student written assignment that is handed in to be graded by the teacher.)

2. Three aspects of investigation have been exemplified here. There are of course many other examples that might have been provided under these headings. (Interested readers will find many of these in the publications section of my website: http://wwwstaff.murdoch.edu.au/~kissane and many others in the 'Activities' of my calculator books.) But are there other *headings*?

3. I have consciously not mentioned programming as an investigative activity of significance to graphics calculators, mainly for space reasons. However, a significant aspect of graphics calculators is their programmability. This is explored a bit on my website at
http://wwwstaff.murdoch.edu.au/~kissane/gcprograms.htm
In what ways does programmability of calculators give rise to new opportunities for investigation?

4. What kinds of stimuli will be most successful in engaging students in investigative tasks with their calculators? Readers may like to make use of one or two of the specific suggestions in the paper and report on the consequences of exploring them with their students.

5. How can we avoid the 'electronic algorithm'? That is, replacing the previous algorithms that have plagued mathematics education with new, calculator versions. While the motive to offer students 'help' of this kind seems to have external examinations in mind, it seems very likely that such uses of calculators are likely to do more harm than good. Do others agree that this clearly anti-investigative use of graphics calculators is a problem? What should we do about it?

Source: This paper was presented at the Australian Association of Mathematics Teachers Virtual Conference, 1999, and is published on the associated CD-ROM.