

The graphics calculator and the curriculum: The case of probability*

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A graphics calculator may influence the curriculum in three kinds of ways: offering new opportunities for teaching and learning, providing new ways of performing mathematical procedures, and suggesting changes of emphasis and content in the curriculum. Illustrations of these for the senior school probability curriculum are suggested, focussing on simulation, probability distributions and stochastic modelling respectively. The Casio cfx-9850G calculator is used to explore these examples.

It is perhaps understandable that most interest in graphics calculators to date seems to have focussed on their use in dealing with functions, particularly graphs of functions. However, a graphics calculator is a versatile piece of technology that can be used to support a number of types of mathematical activity. For example, only four of the 15 chapters in Kissane (1997b) are concerned directly with representing and using graphs of functions; almost 75% of the pages are concerned with other aspects of calculator use. Indeed, the significance of the graphics calculator in schools is related in no small way to the substantial proportion of the mathematics curriculum that may be affected by its availability. A device that is of value to only a small part of the curriculum is not likely to become important enough to either students or teachers to affect what actually happens. The main purpose of this paper is to explore some of the relationships between probability and the graphics calculator, in order to provide a more extensive justification of these sorts of claims.

The history of probability in the school curriculum is relatively short, and generally unfortunate. In most states, aspects of probability have only appeared in the last generation, unlike, say, algebra, geometry, trigonometry and calculus, which have been taught in schools for several generations now. Until quite recently, much of the work has been excessively formal, with a focus in the senior school on the use of an algebra of probabilities. Research, anecdotal evidence and the observations of both teachers and public examiners have generally suggested that students find the formal study of probability rather difficult, and insightful learning seems to be rare. While a formal approach is ultimately essential in the undergraduate years, it is more questionable at the school level, particularly in the lower secondary years. Indeed, in recent times, the status of probability has been both cemented into the curriculum and better understood by the inclusion of a 'Chance and Data' component in the K-10 curriculum in most states. Both *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1990) and *Mathematics – A Curriculum Profile* (Australian Education Council, 1994) have provided a better understanding of the place of probability in the curriculum. A common feature of recent quality curriculum materials is the injection of a practical element into teaching and learning probability, rather than relying exclusively on theoretical analyses. Because of the recent appearance of graphics calculators in schools, however, most resources and curriculum documents make few references to them.

There would seem to be three kinds of ways in which the mathematics curriculum and a calculator might be related to each other. Firstly, the calculator might provide students with opportunities for learning that are not already easily available. Secondly, the calculator may provide alternative means of carrying out mathematical procedures that are more convenient or more efficient in some sense. Thirdly, the calculator may be a catalyst for fresh thinking about the curriculum, to ensure that an appropriate balance

of activities and expectations is achieved. It is not difficult to see examples of each of these three kinds of connections between the calculator as a graphing device and the algebra curriculum, which perhaps explains some of the appeal of graphics calculators. The next three sections of this paper offer some examples for the case of probability. In each case, the excellent Casio cfx-9850G graphics calculator is used throughout as the medium for providing examples.

Simulation

The most significant new opportunities for teaching and learning probability provided by a graphics calculator involve the simulation of data. Although almost all scientific calculators of recent years have contained a command for the generation of (pseudo-) random numbers, these have been rarely used in schools for two main reasons: it has been too difficult to manipulate the random number generated and too tedious to generate enough of them to be useful for educational (or other) purposes. However, all graphics calculators overcome each of these barriers.

Random number commands typically generate random numbers uniformly distributed on the interval (0,1). Figure 1 shows that the relevant command on the Casio-9850 is `Ran#`. A succession of random numbers can be obtained with a single press of the *Execute* key after the command is entered. Unlike a scientific calculator, the results stay on the screen, and do not need to be recorded.

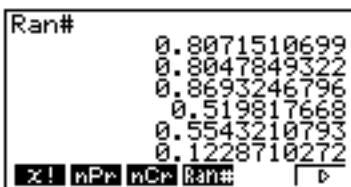


Figure 1: Generation of successive random numbers on (0,1)

It is unusual for a random number in the interval (0,1) to be useful by itself. In most cases, simulated data require some transformation of the random number to suit the purpose at hand. For example, we may want to simulate the rolling of a fair die, so that each of the possibilities in the set $\{1, 2, 3, 4, 5, 6\}$ is equally likely. The linear transformation involved is relatively easy to effect, using a calculator command (`Int`) to find the integer part of a number; the process is described in detail in Kissane (1997a, 1997b). Figure 2 shows one way of constructing such a transformation. Once again, after the command line is entered the first time, each successive press of the *Execute* key provides another random number.



Figure 2: Simulation of rolls of a fair die

Other transformations are convenient for other kinds of purposes. For example, a Bernoulli random variable with probability of success p can be simulated by the calculator command `Int (Ran# + p)`. Figure 3 shows a spatial representation of the transformation involved, with the shaded portions of the number line representing the possible outcomes of executing the commands shown. The command `Int (Ran#+0.4)` thus has a probability of 0.4 of giving the value 1 and a probability of $1 - 0.4 = 0.6$ of

giving a value of 0. In the standard way, a 1 represents a success, and a 0 a failure, for this simulated Bernoulli random variable.

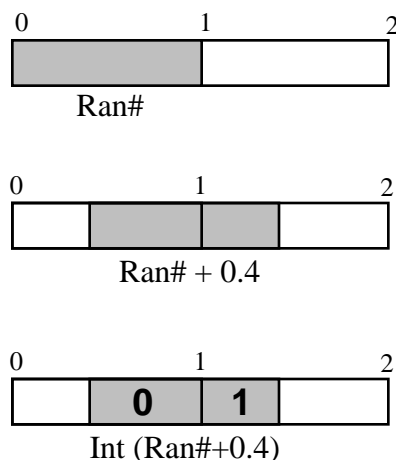


Figure 3: Generating a Bernoulli random variable with probability $p = 0.4$

This mechanism allows for events to be simulated with any probability, while repetitions of this idea allow for binomial random variables to be simulated. The screen in Figure 4 can be interpreted in two ways. It might be regarded as six successive tosses of a biased coin with probability of a head being only 0.4. in which case two heads have appeared.

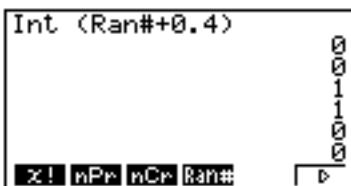


Figure 4: Simulated data using the transformation shown in Figure 3

As the successive simulations are independent of each other, an alternative interpretation of Figure 4 is that it shows a single result (of two heads) resulting from tossing a set of six biased coins, each with $\text{Prob}(\text{Head}) = 0.4$. That is, it shows a single observation (that is, $x = 2$) from a binomially distributed random variable with $n = 6$ and $p = 0.4$.

Simulation of even a small amount of data in these kinds of ways provides an opportunity for students to get a sense of both the unpredictability and the predictability of random phenomena. For example, if each of the students in a class of 30 students simulates six tosses of the biased coin represented in Figure 4, it is likely that most students will get two or three heads, but as many as four or five students may get only a single head, while three or four students are likely to get four heads, and it is even likely that one student will get five out of six heads. The best way to appreciate the unpredictability of random phenomena is to experience them in this kind of way. As far as predictability is concerned, the combined data from a class of 30 students will give close to 40% heads, to provide an experience of long run relative frequency. (All of these predictions are of course probabilistic in nature, and depend critically on the quality of the calculator's random number generator. My experience to date with the Casio-9850 suggests that this confidence is not misplaced.)

These elemental transformations and repetitions can be combined in other ways to suit other purposes. For example, a binomial random variable with $n = 6$ and $p = 0.4$ can be simulated directly by a command consisting of the earlier command repeated six

times over, rather than generating six Bernoulli random variables and counting the number of successes. Figure 5 shows this process, including an abbreviated method of defining the Bernoulli variable (using the graph/table function list, showing in the screen at the left.). Each press of the *Execute* key now simulates the number of heads in six coin tosses, multiplying the amount of data provided to an individual or a class.

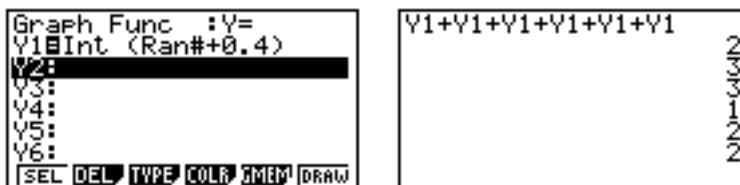


Figure 5: Simulating a binomial random variable with $n = 6$ and $p = 0.4$

These examples illustrate some of the basic capabilities of a graphics calculator for simulation. These can be used to simulate data to understand probabilistic situations or to model real world situations that involve a stochastic element. However, the calculator examples given so far do not provide a great deal more than students can obtain by physical experiments, such as rolling dice or tossing coins, except for a more reliable data generation process that is probably somewhat quieter and quicker. (Incidentally, there is no suggestion here that physical experimentation should be *replaced* by calculator equivalents. Rather, a case might be made for using *both* means of simulating data.) What the calculator *can* provide, that the physical processes cannot, is the possibility of generating a large amount of data and summarising it easily.

Instead of simulating each result by pressing a key, the calculator can be easily set up to generate a set of data, which can then be analysed in the same way that other kinds of statistical data are analysed in the calculator. The Casio-9850 handles such tasks especially well, as does its smaller relative, the Casio-7400. (On some other calculators, it is a little more cumbersome to automatically store a large sample of data of this kind.) Figure 6 shows the calculator set up to generate a table (Y2) consisting of 200 observations from the coin tossing experiment shown in Figure 5. The first four observations in the table are shown in the third screen: 2, 1, 4 and 4.



Figure 6: Simulating 200 binomial observations with $n = 6$ and $p = 0.4$

This table of observations can be transferred (electronically) to the statistics area of the calculator, then summarised and analysed suitably. For example, Figure 7 shows both a histogram of the results as well as some associated statistical data.

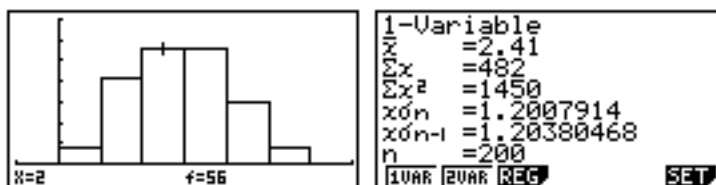


Figure 7: Analysis of 200 simulated data points

In this particular case, there were two and three heads tossed exactly the same number of times (i.e., 56), but a student with this result will see that other students in a class have different results, and she will not get the identical results again next time she repeats the entire experiment (which takes only a minute or so on the calculator). Similarly, the mean number of heads simulated is 2.41 on this occasion, which can also be compared with the findings of other students or repetitions of the experiment by the same student. This kind of simulation has the potential to enrich students' intuitions about probabilistic events considerably. More sophisticated students (say, those who have some experience with the binomial probability distribution) will readily see the connections between the mean score of 2.41 and the values of $n = 6$ and $p = 0.4$; even more sophisticated students can use the whole class data to construct a sampling distribution of the mean scores for the 200 observations, and begin to get a feel for the concept of a sampling distribution.

Probability distributions

The second kind of use of graphics calculators suggested earlier involves carrying out routine procedures. In the case of probability, this may involve evaluating probability distributions, which is of some relevance to senior students of mathematics. In essence, what a graphics calculator might provide is (at least) an alternative to a table book for some distributions, such as the normal, the binomial and the Poisson. It is worth noting in this regard that, apart from statistical computations, the main difference between a scientific calculator and a four-function calculator is that a scientific calculator contains electronic versions of the standard table books, thus avoiding the need to look up trigonometric, logarithmic and exponential tables. Some scientific calculators also contain normal probability distribution values in addition to these.

Graphics calculator manufacturers have not yet reached consensus on which distributions to incorporate into their various models. Until recently, many calculators had no distributional information stored at all, in fact. (It seems likely that the Advanced Placement examinations in Statistics in the USA may well be the major determining influence on this issue.) To illustrate the concept, the Casio-9850, which provides normal distribution probabilities, sits between the two extremes of providing no distributions, and trying to provide all that might be needed. Figure 8 shows that both numerical and graphical values for the standard normal distribution can be obtained, with the calculator providing a choice of three possible areas under the curve.

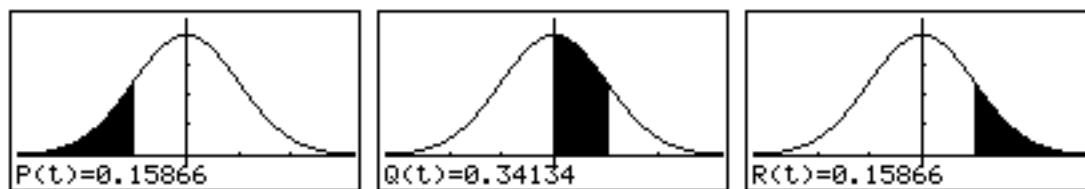


Figure 8: Calculator versions of $\text{Prob}(z < -1)$, $\text{Prob}(0 < z < 1)$ and $\text{Prob}(z > 1)$

The calculator provides some advantages over the standard table look up procedure in this case. In the first place, the graph allows a user to verify visually that the correct probability is being calculated; a common error with tables is to either look up the incorrect value, or to deal with the value inappropriately once found (e.g., to subtract it from 1.) In the second place, the calculator provides direct access to values that are not normally tabulated, and hence need to be calculated from symmetry considerations. Thus, it is rare (but not unknown) for normal probability tables to tabulate both negative and positive values for z .

Practical questions frequently demand more of students than merely looking up a table; for example, differences between tabulated values are sought, requiring some manipulation of values by students. However, a graphics calculator that has a

distribution command available allows for *direct* computation of the required values. Thus Figure 9 shows how to evaluate $\text{Prob}(-0.5 < z < 1.5) = 0.62465$, $\text{Prob}(|z| > 1.8) = \text{Prob}(z < -1.8) + \text{Prob}(z > 1.8) = 0.07186$ and $\text{Prob}(z > -0.7) = 0.75804$. In the past, many students have found such tasks difficult, because the available means of evaluation, using tables and symmetry properties of the normal distribution, is too complicated.

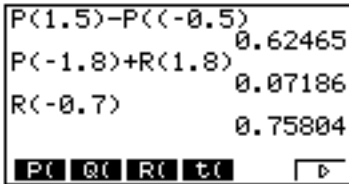


Figure 9: Direct calculation of normal distribution probabilities

As well as direct evaluations of individual probabilities, a graphics calculator can be used to generate a table of probabilities, using the table functions and distribution functions in concert. Figure 10 shows an example of this sort of facility, for the particular case of a normal distribution with mean $\mu = 50$ and standard deviation $\sigma = 10$. In this case, table values are set at every integer from 45 to 55, but of course finer or coarser steps can be chosen to suit a particular purpose.

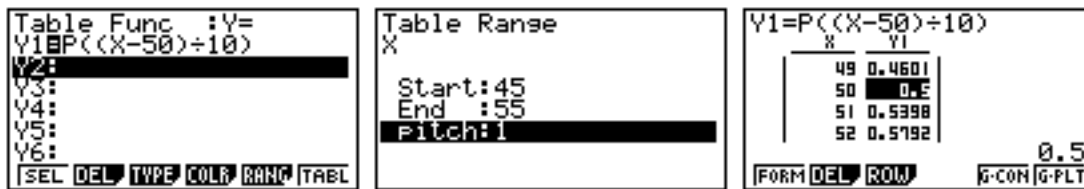


Figure 10: Tabulating a normal distribution with $\mu = 50$ and $\sigma = 10$.

Since looking up tables, even electronic tables, can be tedious, the versatility of the graphics calculator can be exploited to provide probability distribution information in other ways as well. For example, a one line program (Kissane, 1996) can be written to allow repeated direct input of X values, and calculation of the associated normal probability $\text{Prob}(x < X)$, using the same distribution as in Figure 11. Each press of the *Execute* key allows for the next X value to be input, taking advantage of the Casio's input command (symbolised by the question mark). In this way, various levels of precision are easily accommodated, such as $\text{Prob}(x < 52)$ and $\text{Prob}(x < 49.6)$.

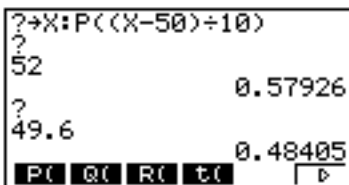


Figure 11: A one line program for calculating normal probabilities ($\mu = 50$, $\sigma = 10$)

These ideas are not restricted to the normal distribution, but additional work is needed to deal with distributions that are not available routinely on a particular calculator. In some cases (such as the normal distribution itself), this may require the use of a numerical integration facility, while in other cases, such as the Poisson and binomial distributions, direct calculations are possible.



Figure 11: Table of values for the binomial distribution with $n = 30$ and $p = 0.3$

To illustrate this point, Figure 11 shows a table of values for the binomial distribution with $n = 30$ and $p = 0.3$. Such a table is beyond the range of most table books. The function tabulated, which does not all appear on the screen shown in Figure 11, is $\text{Prob}(x) = {}^{30}C_x \times 0.3^x \times 0.7^{30-x}$. The highlighted value shows that $\text{Prob}(x = 10) = 0.1416$ for this distribution.

Stochastic modelling

Finally, the third kind of relationship between a graphics calculator and the curriculum involves reconsidering the balance of the curriculum in the light of available technology. Knee jerk responses to such questions are clearly inappropriate; it is no more sensible to inject new ideas into the curriculum solely because a calculator can deal with them than it is to prohibit calculators from being used just because they may interfere in some sense with the traditional curriculum. Such issues need more careful thinking than this, and require some informal experimentation in schools as well.

Mathematical modelling techniques are always subject to the constraints of available mathematics and technology. In the case of stochastic modelling, the easy means of manipulating matrices on graphics calculators increases the options available, and it is of interest that each of three modern curricula (the Nuffield Advanced Mathematics Project and the School Mathematics Project, 16-19 in the UK and The University of Chicago School Mathematics Project in the USA) have considered this matter and decided to take advantage of the new possibilities.

For illustrative purposes, suppose that a weather forecaster estimates that if it is fine one day, then the probability that it will be fine the next day in a certain place is 0.8. Suppose further that, if it is raining one day, the probability of rain the next day is 0.3. These conditional probabilities provide a model to predict likely weather patterns, on the rather simplistic assumption that the weather can be described as either 'rain' or 'fine' each day. The model can be represented in various ways. A matrix representation is the most convenient, as shown in Figure 12, which also includes a calculator version of the 2×2 matrix T , which is sometimes described as a *transition matrix*.

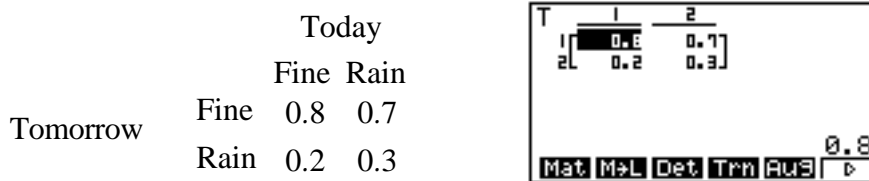


Figure 12: Matrix representations of a stochastic model

The weather on a particular day can be modelled with a column vector, A , a 2×1 matrix. Suppose that it actually *is* fine today. Then the weather today can be represented by the matrix shown in the first screen in Figure 13. Tomorrow's weather can be predicted by evaluating the matrix product TA , the next day's weather can be predicted by calculating T^2A , the weather for the following day can be predicted to be T^3A , and so on. It is not difficult to perform these kinds of calculations on calculator with matrix capabilities, but it is rather complicated to do so by hand.

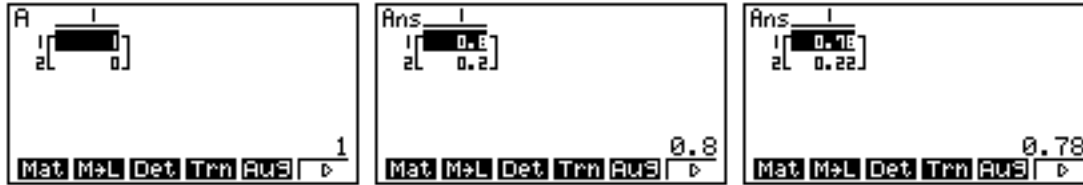


Figure 13: Matrices A , TA and T^2A , used to predict the weather

Longer term predictions can be made using this model and the calculator's capacity to multiply matrices efficiently. For example, Figure 14 shows the predictions for the weather ten days after a fine day.

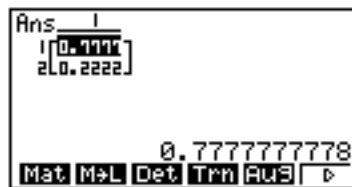


Figure 14: Predicted weather ten days after a fine day

Under the model assumptions, the probability of a fine day is about $7/9$, while the probability of rain is the complement, $2/9$. In this case, the predicted weather ten days after a *rainy* day is essentially the same, which may come as a surprise to students. Indeed, the long term predictions for the weather are the same with this model, regardless of today's weather, which provides an illuminating insight into the model.

This is a simplistic example, but these same kinds of operations can be used to predict population changes, based on models that predict different sorts of effects for different kinds of populations (such as those of different ages). These so-called *Leslie matrices* are discussed briefly in Kissane (1997b). Realistic population growth models, as well as other physical phenomena, can be explored using the technology of the graphics calculator. Theoretical work of this kind, involving the formal study of Markov processes, is not appropriate for senior school, but a case might be made for close consideration of the merits of numerical work such as that described here.

Conclusion

The graphics calculator has the potential to influence the mathematics curriculum and teaching in three main ways. It can offer new teaching and learning opportunities, it can provide convenient computational alternatives and it can render previously inaccessible mathematics more appropriate for senior school. Examples of each of these three are described and discussed for the particular case of probability.

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