Symbolic manipulation on a TI-92: New threats or hidden treasures?*

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The availability of hand held devices that can undertake symbolic manipulation is a recent phenomenon, potentially of great significance for both the algebra and calculus curriculum in the secondary and lower undergraduate years. The significance to date of symbolic manipulation for mathematics is described, and parallels drawn with the significance of arithmetic skills for the primary school. It is suggested that, while symbolic manipulation is central to mathematics, many students develop only a restricted competence with the associated mathematical ideas. The Texas Instruments TI-92 is used to suggest some potential beneficial uses of technology that involves symbolic manipulation.

During the last generation, school mathematics curriculum and practice have been substantially affected by the proliferation of inexpensive devices for dealing with arithmetic. Around twenty years ago, hand calculators began to be affordable and available in schools; today, they are virtually ubiquitous. Over the same period of time, the microcomputer has developed from being a gleam in the eye of rich kids playing in their Californian garages to almost reaching the same kind of status as the home stereo set, at least in affluent countries like Australia; yet the microcomputer has affected mathematics curriculum and practice remarkably little. While each of these evolutions has been unfolding, there have been equally remarkable changes in the capacity of electronic technologies to deal with algebra and calculus in a convincing and convenient way, although until quite recently this has been less obvious to most people.

Mainframe computers of the 1970's were used to run large symbolic manipulation systems, but these were certainly not in widespread use. They were followed by microcomputer versions, the first of which was *Mu-Math*, which was even available for the Apple II series of microcomputers. By the mid-1980's, *Mu-Math* was 'remaindered' by its publishers and replaced by the much more powerful *Derive*. At around the same time, *Maple* was developed, and other examples followed, including the very powerful *Mathematica*. Although this kind of software has made recent, and generally small, impacts on some parts of tertiary education, it has remained too expensive, and required too much computer technology to operate it for many schools to take much notice of it.

Now the processing power of the microcomputer, the convenience and affordability of the hand calculator and the symbolic manipulation capabilities of large software packages are all bundled into a single device. The first such examples of this new iteration were probably Hewlett Packard's HP-28 and HP-48 series of calculators; the next was the Texas Instruments' TI-92, a detailed description of which is provided by Kissane (1996). But these will certainly not be the last examples. In this paper, we explore some of the potential links between mathematics and personal technologies that allow symbolic manipulation to be performed.

The significance of symbolic manipulation

It would not be an exaggeration to suggest that symbolic manipulation is seen as no less central to the secondary school mathematics curriculum than arithmetic is to the primary curriculum. Many students, and many teachers, regard the development of a fluency with symbolic manipulation as the main sign of progress in mathematics, and the most significant indicator of the likely prospects of students continuing to make further progress. Indeed, at a practical level, these views would seem to be entirely justified, since most current sequences of school mathematics courses have increasingly high entry fees as far as symbolic manipulation is concerned, and a surprisingly high proportion of time within advanced courses is spent on further developing symbolic manipulation. However, not everyone agrees that this is a desirable state of affairs. For example, Robert Davis was less than complimentary about such an emphasis:

> At one extreme, we have the most familiar type of course, where the student is asked to master rituals for manipulating symbols written on paper. The topics in such a course have names like “removing parentheses,” “changing signs,” “collecting like terms,” “simplifying,” and so on. It should be immediately clear that a course of this type, focussing mainly on meaningless notation, would be entirely inappropriate for elementary school children; many of us would argue that this type of course, although exceedingly common, is in fact inappropriate for all students. (1989, p 268)

Recent curriculum development projects have also been a little more circumspect regarding the development of symbolic manipulation skills in schools. For example, both *Investigating Change* (Barnes, 1992) and *Access to Algebra* (Lowe et al, 1993-94) consciously placed new kinds of emphasis and sequencing of symbolic manipulation. Each of these two sets of curriculum materials has emphasised the importance of the concepts involved in algebra and calculus, and provided students with opportunities to develop a thorough understanding in these before expecting a high level of 'speed and accuracy' in symbolic manipulation.

A fluency with symbolic manipulation is certainly vital for continued progress in mathematics. Indeed, it might be suggested that students do not 'really understand' algebra and calculus until they have become free of the shackles of the symbolic forms in which they are represented. However, it is not easy to recognise this state of mind, and certainly not easy to bring it about.

The most important sense in which symbolic manipulation is central to mathematics is that it provides access to general cases and exact solutions, rather than merely specific cases and approximate solutions. Until fairly recently, this distinction was difficult to illustrate, since the only available means of solving many problems involved symbolic manipulation. For example, except for some special cases such as those concerned with parabolas, the relative maximum of a function on an interval could only be found to reasonable accuracy through using differential calculus. In turn, the only way of dealing with differential calculus was through extensive experience with symbolic manipulation (to find derivative functions). Nowadays, there are ready alternatives to finding a relative maximum of a function on an interval, through the use of a graphics calculator, for example. Although the solution is not exact, it is sufficiently accurate for any practical purpose. The approximate solution is accessible to many students some years before the symbolic and general solution is accessible.

In the same way, students can these days produce a good sketch of a graph of a function on a calculator very quickly, rather than requiring years of algebra and then calculus to reach the same endpoint, through a process of finding relative extrema, asymptotes and so on. The traditional route provides enough exact information for someone to represent a curve well, whilst the graphics calculator route (usually) provides a much better graph than can be drawn by hand, much more quickly than can be derived from analytic study of the function using calculus. Both approaches have their place in the mathematical repertoire of a modern student.

At an even less sophisticated level, finding the height of a cylinder with a given volume and radius can also be dealt with either generally or specifically. The general solution involves some rearrangement of the formula $V = \pi r^2 h$ to produce a new formula, into which the values can then be substituted. However, a specific solution can be found, by first substituting into the formula and then solving the resulting equation.

The mathematics curriculum in schools and the early undergraduate years has generally focussed on problems for which symbolic solutions are available, in part because there has been no alternative. For example, we have only dealt with cubic equations that can be readily solved through factorisation. Now there is an alternative.

A change of emphasis

A person fluent with a particular algebraic concept or principle has three distinctive and defining characteristics:

i) they can decide when it might be useful to use this concept or principle;
ii) they have the symbolic manipulation skills to do it correctly;
iii) they know what the significance of the end-product is.

We suggest that too much time has been spent traditionally on the second of these, and too little time on the first and the last. The significance of symbolic manipulation capabilities supported by technology may be that the balance of time may be shifted a little. The parallel with arithmetic in the primary school is striking. It is one thing to know how to carry out, say, long multiplication correctly. But it is quite a different matter to know when it is a good idea to multiply two numbers together, and what the relationships between the results and the starting numbers are (in this case, concerned with both division and multiplication). The four-function calculator is finally being recognised as a device that is demanding that we pay more attention to all three aspects of multiplication, redressing old imbalances.

We encounter a problem when we try to find out how well students understand a particular algebraic concept or principle. We illustrate the problem for the particular case of the difference of two squares. On the one hand, we would not like to think that a student 'understood' the associated principle until she regarded the following as 'the same' in some sense:

$$a^2 - b^2 \qquad a^2 - c^2 \qquad x^2 - y^2 \qquad A^2 - B^2$$

Indeed, we would probably also want to be confident that students recognised each of the following expressions as also 'the same' in this sense, before we would be confident saying that she understood the idea.

$$4a^2 - 9 \qquad a^2b^2 - c^2 \qquad x^4 - y^6 \qquad (a + b)^2 - (a - b)^2 \qquad \sin^2\theta - \cos^2\theta$$

The principle of the difference of two squares is about the *form* of an expression and our interest is in whether or not students recognise the form and use it to advantage. At present, our only window into seeing how well students have grasped this idea is to inspect their symbolic manipulation competence. The source of our problem is that students can be taught to carry out the symbolic manipulations directly (perhaps even as an assortment of special cases), and yet not learn the most important characteristic that each has in common, or even why this is worth paying attention to in the first place.

In the case of a difference of two squares, many students have developed the symbolic manipulation skills (often after a great deal of practice), but yet do not apply them unless prompted by a command such as "factorise". Alternatively, they may apply them when there is no advantage in doing so:

$$x^2 - 9 = x$$
$$(x - 3)(x + 3) = x$$

Similarly, many students do not have a sense that a difference of two squares provides them with a pair of *factors* of an expression, which can be quite useful when trying to simplify a rational expression.

We suspect that, in the quest to develop symbolic manipulation skills in students, we too often feel obliged to devote a lot of students' time directly to the manipulative skills themselves, and too little time on the surrounding contexts. One potential advantage of the use of symbolic manipulation technologies is that we might change this emphasis to an extent. This is not unlike the situation in the primary school as far as arithmetic is concerned, as suggested in the next section.

An analogy with the primary school

As suggested above, fluent use of arithmetic involves much more than speed and accuracy with computation. Similarly, there are many important concepts that need to be developed before a fluency with symbolic manipulation is worth developing. The *National Mathematics Profiles* do a good job of identifying many of these, in the algebra strand, under the three substrands of *Expressing generality*, *Functions* and *Equations and inequalities*. The relevant concepts include (but are not restricted to) the following: expression, formula, factor, equivalent expression, identity, equation, integration, differentiation, limit.

These are all rich concepts, as can be seen by closely considering any of them. Take the case of equations, for example. As with the example of multiplication described above, coming to terms with equations in algebra requires firstly that students develop an understanding of what equations are, where they come from, why we might be interested in solving them and how to produce them. Only then does it make much sense to develop techniques for solving them via a process of symbolic manipulation. Solving equations requires a good feel for what a solution is, and demands considerable interpretive skills of students to interpret the results of their labours. Our impression, echoed by the observations of Davis above, is that, for a variety of reasons, many students have interpreted their school experience with equations as mainly concerned with 'mastering rituals', and less concerned with the conceptual bases of the rituals.

Of course, in practice, the development of a concept and the development of the associated symbolic manipulation skills do not happen in strict temporal sequence. Frequently, the two develop over the same period of time. All too often in the past, however, students have developed symbolic manipulation skills without developing the concepts associated with them. This can happen because there is an urgent need to develop the skills, and it is hoped that the concept will develop as a consequence.

It is significant that devices such as Texas Instruments' TI-92 can now handle most of the symbolic manipulation associated with the important concepts of algebra and calculus. The pressing question for mathematics education now concerns the appropriate response to this development. The question is much too significant to be ignored. We suggest that it might be helpful to think about the effects of such a device in algebra and calculus, conscious of the effects on primary school arithmetic of the four-function calculator. Reasoning by analogy is often a hazardous undertaking, but it may be profitable to think about the use of devices such as the TI-92 in a similar way in which the four-function calculator has been thought about in the primary school. These include the following, not all of which are regarded as benign uses for technology:

Prohibition: For some children, calculators are prohibited. In some cases, it is a general prohibition, while in other cases, it is more particular. (For example, they cannot be used in the test.) To date, using technology for symbolic manipulation has been prohibited in schools, partly because it is too expensive. Equity issues associated with examinations are obvious if only some students have access.

Checking: Children are allowed, or even encouraged, to use their calculator to check their work. It is still expected that they will do their work without the calculator, and they may even be denied access to a calculator until quite late in their school career. Although this practice is hard to defend, it still seems quite common. It is conceivable that similar uses for a TI-92 might be contemplated.

Substitution: Usually (but not always) without sanction from their teacher, children might use their calculators to do arithmetic instead of learning alternative ways. The analogy with a symbolic manipulation device is easy to make. At the least, substitution is rendered possible by technology. If we want to prohibit students from doing this, we need to be able to defend our policy. To dissuade students from substituting inappropriately with technology, we will need better arguments than, "the batteries might go flat" or "you won't really understand what you are doing unless you do it the long way by hand".

Simultaneous use: Developing arithmetic competence may take place in an environment in which paper-and-pencil, mental and calculator work all happen together. At issue is the locus of control: whether it is the teacher or the children who decide which kinds of technologies to use at a particular time. It seems likely that this sort of environment is the most likely one for students to develop some discretion about when to use a calculator and when not to use a calculator, although some explicit attention needs to be paid to helping them make such decisions.

Complexity: Children may be encouraged to use their calculators for complicated situations, such as those involving large numbers or many successive calculations. In the analogous way, a TI-92 might be used when a situation demands particularly complicated algebraic manipulations or especially difficult integrals, for which general solutions are sought.

These categories of use are suggested as a first step in considering appropriate use of these new forms of technology. Now that we have a generation of experience with less sophisticated technology, we may be able to learn something from it when thinking about more powerful technologies. The rest of this paper offers some examples of possible uses of symbolic manipulation devices in schools and the early undergraduate years. Clearly, the examples chosen are constrained by space, and we do not regard them as being definitive, but rather offer them as suggestive of some ways forward.

Some examples of uses

The two screens shown in Figure 1 provide a glimpse into some of the powerful symbolic manipulation capabilities of the TI-92. Developing an understanding of the relationships between factors of a polynomial function and its roots begins with quadratic functions in the middle of the secondary school, but rarely proceeds much further because of the tedium associated with finding the factors or finding the roots. Such links are critical to solving polynomial equations, and have clear connections with graphs of functions.

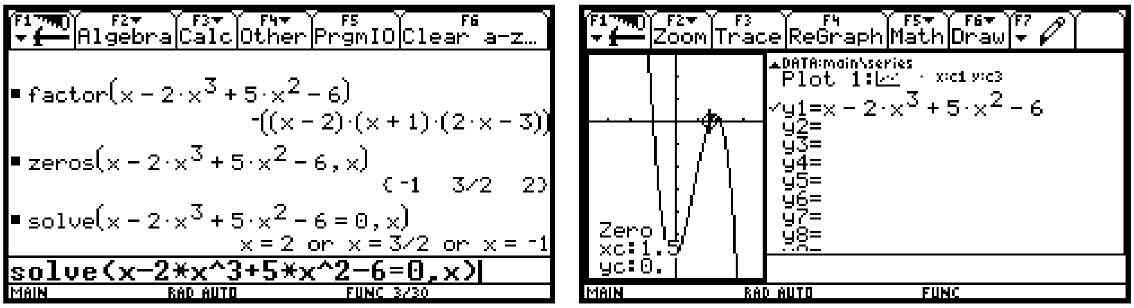

Figure 1: Relationships between roots, factors, solutions and a graph

For many students, we suggest that the time required to master the symbolic manipulation required for factorisation and (exact) solution of polynomial equations restricts their opportunities to understand completely the connections illustrated in Figure 1. Knowing thoroughly about the connections may help students decide what kinds of manipulations are needed, regardless of whether they ultimately decide to carry out the manipulations by hand or by machine.

As noted above, many students have difficulty with transposition of formulae, which provide access to general solutions of equations rather than specific solutions. In many cases, the TI-92 can be used to obtain a transposed formula (by 'solving' a formula) as shown in the screen at left in Figure 2. In other cases, the TI-92 provides a general solution that is difficult to understand or awkward to express, however, as in the first solution of the formula to obtain an expression for *r* on the right screen of Figure 2.

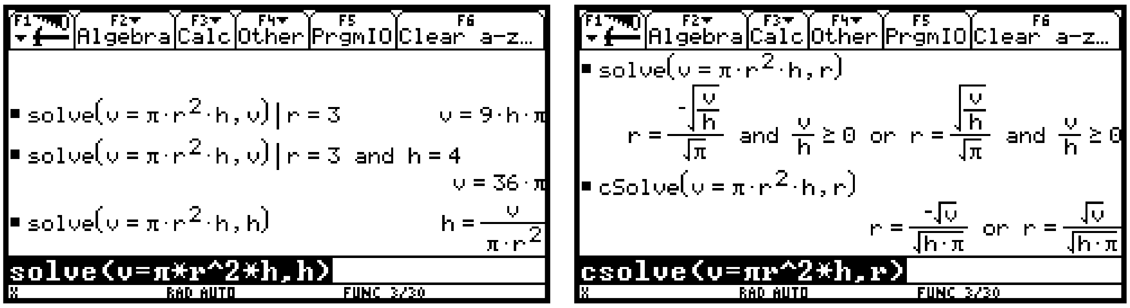

Figure 2: Dealing with a formula

The second solution on the right of Figure 2 uses a complex (number) command *cSolve* to bypass this problem. The example illustrates that using a symbolic manipulation device still requires students to understand what they are doing, and may even need to develop new ideas, in much the same way that young children may encounter negative numbers and decimals prematurely by having access to a four-function calculator.

When students first encounter differentiation, it is quite common in calculus curricula for them to learn about first principles definitions of the derivative of a function. While common, it is by no means universal these days. The conceptual problems of dealing with limits are considerable, and some are content with leaving a formal definition of the derivative (as the limiting value of a slope) until fairly late, relying on less formal ideas to develop the concept better. The idea of local linearity is especially powerful, and available to students with a graphics calculator.

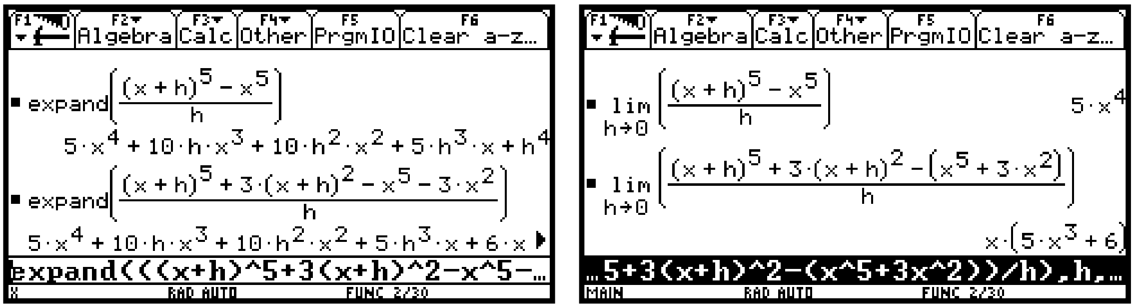

Figure 3: Finding derivatives from first principles

Regardless of the formality of the treatment of limits, work of this kind quickly becomes excessively complicated in terms of algebraic manipulations, so that many students are barely able to cope with much other than the least complicated examples, such as the quadratic function, $f(x) = x^2$. Figure 3 shows that some of these sorts of complicated manipulations can be left to the machine, with the hope that students might be able to focus on the ideas involved, and the general patterns of results a little more easily. We hope that they might see the (differential) forest for the (algebraic) trees. The manipulations associated with the function $f(x) = x^5 + 3x^2$ are also shown in Figure 3. We would hope that students would be able to construct the expressions for themselves (and need to be able to do so, in order to instruct the TI-92 what to do), and hope too that they would not be surprised at the (factorised) result in the right screen. We do not think that much is lost by leaving the details of moving from the idea of gradient or limiting gradient to the general result via machine – except perhaps time-consuming frustration.

The screens shown in Figure 4 also illustrate the use of a TI-92 for differentiation, but this time obtaining the derivatives directly. The way in which the TI-92 operates allows users to quickly edit commands, and thus produce many examples in a short space of time. The potential benefit of this is that students may see for themselves the connections between a polynomial and its derivatives.

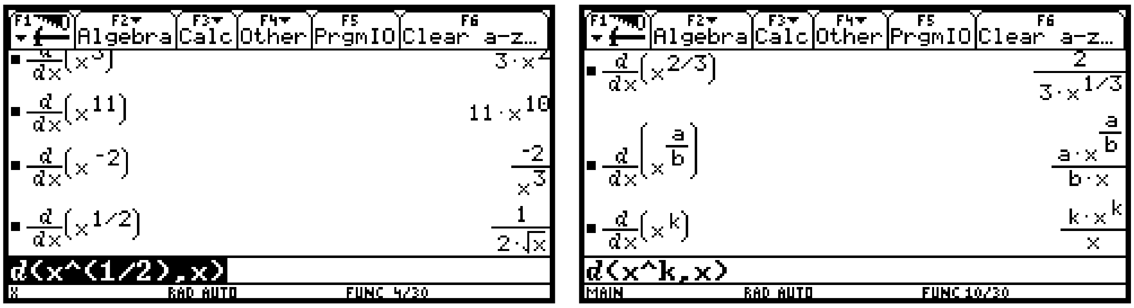

Figure 4: Derivatives of $f(x) = x^n$, for n an integer or a fraction and for general n

Figure 4 also shows the general cases of $f(x) = x^n$ and $f(x) = x^{a/b}$. As the screen on the right shows, the TI-92 at times presents information in a simplified or rearranged form, requiring the user to thoughtfully interpret the screen. Rather than regard this as a defect of the technology, we would prefer that it be seen as a way of encouraging reflective use by students.

As for differentiation, much typical student work related to integration is cluttered with algebraic manipulation. Since the TI-92 will usually simplify results, there are risks that insights might be missed if too much is left to the machine. But, as the screen at the left of Figure 5 shows, the algebraic capabilities of the device can also be used to demystify as well. The possibly unexpected result at the bottom of the screen can be seen as a consequence of the algebraic rearrangement in the middle of the screen.

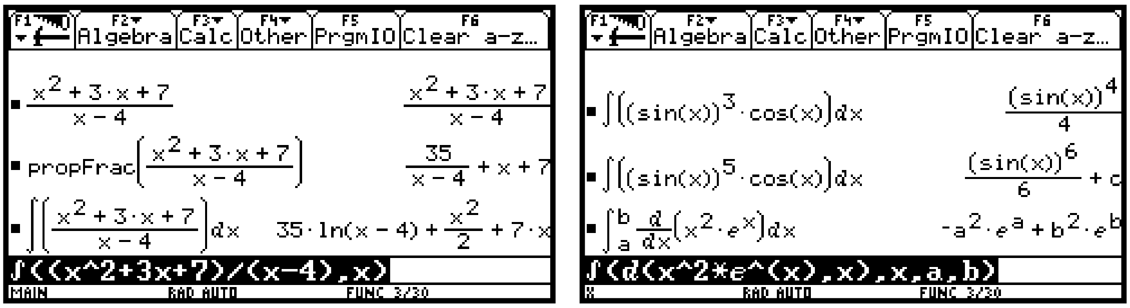

Figure 5: Aspects of integration

Figure 5 also illustrates that access to a TI-92 might offer students some insights that support other aspects of their thinking about integration. While integration by substitution is an important idea that students need to develop some facility with, seeing some integrals such as those involving circular functions above may help them to deepen their insights into the process. Similarly, student understanding of the Fundamental Theorem of Calculus may be enhanced by the TI-92, by finding integrals of derivatives, focussing on the results rather than the means of getting them.

As well as such obviously didactical uses, a symbolic manipulation device also allows students new kinds of opportunities for doing mathematical work, including the investigation of new mathematical situations. A TI-92 may provide access to hidden treasures of mathematics that were previously camouflaged in a morass of symbols. For example, if investigating the turning points of cubic functions, students are likely to proceed by looking at a number of particular cases, and trying to generalise from these.

However, an important aspect of the power of symbolic manipulation, as suggested earlier, is its capacity to express and deal with the general situation as well as the specific. Figure 6 provides an illustration of this sort of generality.

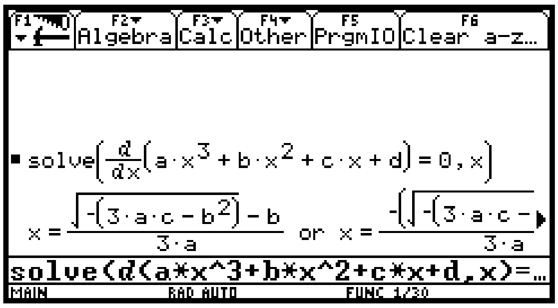

Figure 6: Finding the turning points of a general cubic function

The command in the screen shown in Figure 6 instructs the TI-92 to find turning points by solving the equation that results by setting the first derivative to zero. (Not all such solutions lead to turning points, of course.) The TI-92 handles comfortably the general case, and obligingly simplifies the solutions of the resulting equation into an appealing form. Although a competent student could deal with such a problem, very few have bothered to do so, in our experience, because the algebraic manipulations seem to be too daunting. As with complicated arithmetical calculations, some sorts of things are better left to machines which are less likely to get bored, make a small (but critical) slip or produce an incorrect result. The turning points are expressed by the TI-92 in a slightly idiosyncratic form, but are readily expressed in a more familiar way:

$$x = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$$

Some interesting mathematics can be encountered by using these expressions and imagining, or checking by substitution, what happens for various values of the coefficients *a*, *b*, *c* and *d*. For example, the significance of the fact that *d* is not involved in the solutions, that *a* cannot be zero and that the solutions have an element of symmetry are all worthy of consideration. This final example illustrates that a symbolic manipulation capability such as the TI-92 may help students to see that there are surprising results in mathematics, if we learn to look hard enough in the right direction.

Conclusion

After the invention of the four function calculator, arithmetic could never be quite the same again, even if the calculator were not used extensively. The same kind of observation might be made regarding symbolic manipulation and the familiar algebra and calculus trunk of school mathematics.

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Kissane, B., Bradley, J. & Kemp, M. 1997, Symbolic manipulation on a TI-92: New threat or hidden treasures? In N. Scott & H. Hollingsworth (Eds), *Mathematics: Creating the future*, Melbourne, Australian Association of Mathematics Teachers, 388-396.